# 1.2 Axiomatic Systems in Propositional Logic

## 1.2.1 Description

Axiomatic systems are the oldest and simplest to describe (but not to use!) type of deductive systems. The probably first prototype of an axiomatic system can be found in Euclid's *Elements* which present a systematic development of elementary geometry, based on several simple assumptions about points and lines (such as "Every two different points determine exactly one line", "For every line there is a point not belonging to that line", etc.). Using these and (informal) logical reasoning, other geometric facts are derived, and thus the entire body of Euclidean geometry is eventually built. But the *logical* concept of axiomatic system was introduced and formalized only at the beginning of the 20th century by the German mathematician David Hilbert (who also essentially re-wrote much of Euclid's *Elements* in a modern and rigorous style) so that they are also known as *Hilbert-style systems*.
Axiomatic systems use very few and simple rules of inference, but are based on *axioms*. Here we will build a hierarchy of axiomatic systems for the propositional logic, by gradually adding axioms for the logical connectives.
If we assume that the only logical connectives are $\rightarrow$ and $\neg$ and all others are definable in terms of them, then the axiomatic system comprises the following axioms and rules:

*Axioms schemes:*

($\rightarrow$ 1) $A \rightarrow (B \rightarrow A)$;

($\rightarrow$ 2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$;

($\rightarrow$ 3) $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$.

The only *rule of inference* is **Modus ponens:**

$$\frac{A, A \rightarrow B}{B}.$$

We will denote this system by $\mathbf{H}(\rightarrow, \neg)$.

If we consider the conjunction as a primitive, too, rather than definable connective, then it suffices to add the following axiom schemes for it:

($\wedge$1) $(A \wedge B) \rightarrow A$;

($\wedge$2) $(A \wedge B) \rightarrow B$;

($\wedge$3) $(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B \wedge C))$.

Likewise, the following axiom schemes suffice for the disjunction:

($\vee$1) $A \rightarrow A \vee B$;

($\vee$2) $B \rightarrow A \vee B$;

($\vee$3) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$.

The axiomatic system comprising all instances of these axiom schemes and the rule Modus Ponens will be called simply $\mathbf{H}$.

### 1.2.2 Some derivations

A formula $A$ which can be derived by using the axioms and applying successively Modus Ponens is said to be *derivable in* **H**, or a *theorem of* **H**, which we will denote by $\vdash_{\mathbf{H}} A$.
Using **H** one can derive logical consequences, too. In order to derive "If $A_1, \ldots, A_n$ then $B$" we add the premises $A_1, \ldots, A_n$ to the set of axioms and try to derive $B$. If we succeed, we say that $B$ is *derivable in* ***H*** *from the assumptions* $A_1, \ldots, A_n$, denoted $A_1, \ldots, A_n \vdash_{\mathbf{H}} B$. In particular, $\varnothing \vdash_{\mathbf{H}} B$ means simply $\vdash_{\mathbf{H}} B$.

**Example 1** $\vdash_{\mathbf{H}} (p \wedge (p \rightarrow q)) \rightarrow q$*:*

1. $\vdash_{\mathbf{H}} (p \wedge (p \rightarrow q)) \rightarrow p$, *by Axiom* $(\wedge 1)$;
2. $\vdash_{\mathbf{H}} (p \wedge (p \rightarrow q)) \rightarrow (p \rightarrow q)$, *by Axiom* $(\wedge 2)$;
3. $\vdash_{\mathbf{H}} ((p \wedge (p \rightarrow q)) \rightarrow (p \rightarrow q)) \rightarrow (((p \wedge (p \rightarrow q)) \rightarrow p) \rightarrow ((p \wedge (p \rightarrow q)) \rightarrow q))$, *by Axiom* $(\rightarrow 2)$;
4. $\vdash_{\mathbf{H}} (p \wedge (p \rightarrow q)) \rightarrow p) \rightarrow ((p \wedge (p \rightarrow q)) \rightarrow q)$, *by 2,3 and Modus Ponens;*
5. $\vdash_{\mathbf{H}} (p \wedge (p \rightarrow q)) \rightarrow q$, *by 1,4 and Modus Ponens.*

One can check that all axioms of **H** are tautologies and therefore, since the rule Modus Ponens is valid, **H** can only derive tautologies when using these axioms as premises. By the same argument, **H** can only derive *valid* logical consequences. Therefore, the axiomatic system **H** is *sound*, or *correct*.
In fact, it can be proved that **H** can derive all valid logical consequences, and in particular, all tautologies, i.e., it is *complete*. Thus, **H** captures precisely the notion of propositional logical consequence.
However, this axiomatic system is not very suitable for practical derivations as is, because even simple cases may require involved derivations. For instance, as a challenge, try deriving the tautology $p \rightarrow p$. Still, the derivations in **H** can be simplified significantly by using the following result which allows us to introduce, and later eliminate auxiliary assumptions in the derivations:

**Theorem 2** *(Deduction Theorem) For any formulae $A$, $B$, and a set of formulae $\Gamma$:*

$$\Gamma \cup \{A\} \vdash_{\mathbf{H}} B \textit{ iff } \Gamma \vdash_{\mathbf{H}} A \rightarrow B.$$

While one direction of this theorem (which?) is an immediate application of Modus Ponens, the proof of other requires a more involved argument which will not be presented here (but see the exercises).
Using the Deduction Theorem the derivation of $\vdash_{\mathbf{H}} p \rightarrow p$ becomes trivial. The derivation in the example above can be simplified, too:

1. $p \wedge (p \rightarrow q) \vdash_{\mathbf{H}} p$, by Axiom $(\wedge 1)$ and the Deduction Theorem;

2. $p \wedge (p \to q) \vdash_{\mathbf{H}} p \to q$ by Axiom ($\wedge$2) and the Deduction Theorem;
3. $p \wedge (p \to q) \vdash_{\mathbf{H}} q$ by 1,2, and Modus Ponens;
4. $\vdash_{\mathbf{H}} (p \wedge (p \to q)) \to q$ by 3 and the Deduction Theorem.

Here is one more example of a derivation in **H** using the Deduction Theorem.

**Example 3** $A, \neg A \vdash_{\mathbf{H}} B$ :

1. $\neg A \vdash_{\mathbf{H}} \neg B \to \neg A$, *by Axiom* $(\to 1)$ *and the Deduction Theorem;*
2. $\neg A \vdash_{\mathbf{H}} (\neg B \to A) \to B)$, *by 1, Axiom* $(\to 3)$ *and Modus Ponens;*
3. $A \vdash_{\mathbf{H}} \neg B \to A$ *by Axiom* $(\to 1)$ *and the Deduction Theorem;*
4. $A, \neg A \vdash_{\mathbf{H}} B$, *by 2,3 and Modus Ponens.*

Note that adding more premises does not affect the validity of derivations, but can only possibly add more derivable formulae. This justifies step 4 above.

**Example 4** $p, \neg p \vdash_{\mathbf{H}} q$*:*

1. $\neg p \vdash_{\mathbf{H}} \neg q \to \neg p$, *by Axiom* $(\to 1)$ *and the Deduction Theorem;*
2. $\neg p \vdash_{\mathbf{H}} (\neg q \to p) \to q$, *by 1, Axiom* $(\to 3)$, *and Modus Ponens;*
3. $p \vdash_{\mathbf{H}} \neg q \to p$, *by Axiom* $(\to 1)$ *and the Deduction Theorem;*
4. $p, \neg p \vdash_{\mathbf{H}} q$, *by 2,3, and Modus Ponens.*

### 1.2.3 Exercises

1. Derive the following in the axiomatic system **H,** using the Deduction Theorem. (Hint: for some of these exercises use the previous ones.)

   (a) $(A \wedge B) \to C \vdash_{\mathbf{H}} A \to (B \to C)$.
   (b) $A \to (B \to C) \vdash_{\mathbf{H}} (A \wedge B) \to C$.
   (c) $\vdash_{\mathbf{H}} A \to ((B \to C) \to ((A \to B) \to (A \to C)))$.
   (d) $\vdash_{\mathbf{H}} ((A \to B) \wedge (B \to C)) \to (A \to C)$
   (e) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
   (f) $(A \to B), (A \to C) \vdash_{\mathbf{H}} A \to (B \wedge C)$
   (g) $\vdash_{\mathbf{H}} ((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
   (h) If $\neg A \vdash_{\mathbf{H}} \neg B$ then $B \vdash_{\mathbf{H}} A$.
   (i) If $\neg A, B \vdash_{\mathbf{H}} \neg B$ then $B \vdash_{\mathbf{H}} A$.
   (j) $\neg\neg A \vdash_{\mathbf{H}} A$.

(k) $A \vdash_{\mathbf{H}} \neg\neg A$.

(l) If $B \vdash_{\mathbf{H}} A$ then $\neg A \vdash_{\mathbf{H}} \neg B$.

(m) $(B \to A) \to ((B \to \neg A) \to \neg B)$.

(n) If $A, B \vdash_{\mathbf{H}} C$ and $A, \neg B \vdash_{\mathbf{H}} C$ then $A \vdash_{\mathbf{H}} C$.

(o) If $\neg A, B \vdash_{\mathbf{H}} A$ then $\neg A \vdash_{\mathbf{H}} \neg B$.

(p) If $A \vdash_{\mathbf{H}} \neg B$ then $A, B \vdash_{\mathbf{H}} C$.

(q) $\vdash_{\mathbf{H}} A \vee \neg A$

(r) $\vdash_{\mathbf{H}} \neg(A \wedge \neg A)$

2. Prove that every theorem of **H** is a tautology.