Advanced Graph Algorithms

Prof. K.R. Chowdhary
Email: kr.chowdhary@iitj.ac.in
Web: http://www.krchowdhary.com

Campus Director,
JIET College of Engineering, Jodhpur

Thursday 27th July, 2017

Connectedness in Graphs

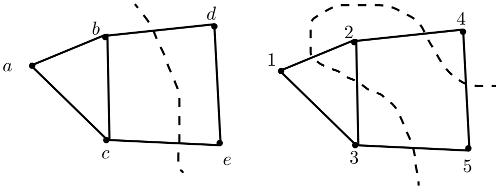
- $G = (V, E)$ is a graph.
- *Connectivity* is one of the basic concepts of graph theory
- A graph is connected when there is a path between every pair of vertices
- In an undirected graph G, two vertices u and v are called *connected* if G contains a path from u to v.
- A directed graph is called *weakly connected* if replacing all of its directed edges with undirected edges produces a connected graph.

A Cut

- **Cut:** A partition of the vertices of a graph into two disjoint subsets. Any cut determines a cut-set, the set of edges that have one endpoint in each subset of the partition. These edges are said to cross the cut.
- In a flow network, an **s-t** cut requires source and sink to be in different subsets, and its cut-set only consists of edges going from source's side to the sink's side. The capacity of an s-t cut is = sum of capacity of each edge in the cut-set.

A Cut...

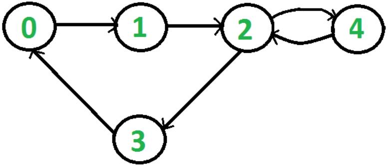
- A cut $C = (S, T)$ is a partition of V into two subsets S and T. $C = (S, T) = \{(u, v) \in E \mid u \in S, v \in T\}$. (min-cut vs max-cut)



- $S = \{a, b, c\}$ and $T = \{d, e\}$, and in other $S = \{1, 3, 4\}$ and $T = \{2, 5\}$

Connectedness in Graphs...

- It is *strongly connected*, *simply strong*



Strongly Connected

- A *cut*, *vertex cut*, or separating set of a connected graph G is a set of vertices whose removal renders G disconnected.
- A *complete graph with n* vertices, denoted K_n, has no vertex cuts at all, but connectivity $k(K_n) = n - 1$.

Connectedness in Graphs...

- Any graph G (complete or not) is said to be k-connected if it contains at least $k+1$ vertices, but does not contain a set of $k-1$ vertices whose removal disconnects the graph; and $k(G)$ is defined as the largest k such that G is k-connected.
- Thus, a connected graph is 1-connected and a biconnected graph is 2-connected.
- Analogous concepts can be defined for edges. In the simple case in which cutting a single, specific edge would disconnect the graph, that edge is called a bridge.

Algorithms for Connectedness

Computational aspects:

The problem of determining whether two vertices in a graph are connected can be solved efficiently using a *search algorithm*,

1. Begin at any arbitrary node of the graph, G
2. Proceed from that node using either depth-first or breadth-first search, counting all nodes reached.
3. Once the graph has been entirely traversed, if the number of nodes counted is equal to the number of nodes of G, the graph is connected; otherwise it is disconnected.

Algorithms for Connectedness: Menger's theorem

- One of the most important facts about connectivity in graphs is *Menger's theorem*, which characterizes the connectivity and edge-connectivity of a graph in terms of the number of independent paths between vertices.
- If u and v are vertices of a graph G, then a collection of paths between u and v is called *independent* if no two of them share a vertex
- The vertex-connectivity statement of Menger's theorem:

  Let G be an undirected graph, x and y two nonadjacent vertices. Then size of the minimum vertex cut for x and y (the minimum number of vertices whose removal disconnects x and y) is equal to the maximum number of pairwise vertex-independent paths from x to y.

Algorithms for Connectedness: Menger's theorem

- The edge-connectivity version of Menger's theorem is as follows:

  Let G be a finite undirected graph and x and y two distinct vertices. Then size of the minimum edge cut for x and y (the minimum number of edges whose removal disconnects x and y) is equal to the maximum number of pairwise edge-independent paths from x to y.
- The number of *mutually independent* paths between u and v is $k(u, v)$, and the number of *mutually edge-independent* paths between u and v is $\lambda(u, v)$.
- By Menger's theorem, for any two vertices u and v in a connected graph G, the numbers $k(u, v)$ and $\lambda(u, v)$ can be determined efficiently using the *max-flow min-cut* algorithm.

Number of connected graphs

- The number of distinct connected labeled graphs with n nodes is:

n	graphs
2	1
3	4
4	38
5	728
6	26704
7	1866256
8	251548592

Min-Cut Max-flow Theorem

- A Network is a directed graph (digraph) $D = (V, A)$ with a capacity function $C : A \rightarrow \mathbb{R}$ assigning arcs to non-negative real values. V can be partitioned into three sets: the sources X, sinks Y, and intermediate I. X, Y must be nonempty.
- To a network we may associate a flow $f : V \rightarrow \mathbb{R}$ assigning arcs to non-negative real values such that $0 \leq f(a) \leq c(a)$ for $a \in A$ and $f_{in} = f_{out}$ for all $v \in I$, where,

$$f_{in}(v) = \sum_{uv \in A} f(uv), \tag{1}$$

and

$$f_{out}(v) = \sum_{vu \in A} f(uv), \tag{2}$$

Min-Cut Max-flow Theorem....

- In other words, the flow over any arc is no more than its capacity,
  The value of flow f, denoted by $val\ f$ is defined as,

$$val\ f = \sum_{x \in X} f_{out}(x) - f_{in}(x) \tag{3}$$

  or, by the above notation, $val\ f = f_{out}(X) - f_{in}(X)$. Given a network, the natural optimization problem is : what is the maximum value attained by any flow?
- A cut $(S, \bar{S})$ in a network is the set of arcs $\{s\bar{s} \in A \mid s \in S, \bar{s} \in \bar{S}\}$ where $X \subseteq S \subseteq V - Y$ and $\bar{S} = V - S$. The capacity of a cut K, denoted as $cap\ K$, is defined as,

$$cap\ K = \sum_{a \in K} c(a) \tag{4}$$

Min-Cut Max-flow Theorem

- Finally, for each cut $K = (S, \bar{S})$ we can define an anticut $\bar{K} = (\bar{S}, S) = \{\bar{s}s \in A \mid s \in S, \bar{s} \in \bar{S}\}$.
- *Max-flow min-cut Theorem.* Maximum of all flow values (i.e., the value of the maximum flow), is equal to the minimum of all cut capacities (i.e., capacity of the minimum cut).

Lemma

Given a network, for any flow f and cut K on the network, $val\ f \leq cap\ K$.

Proof. Let $K = (S, \bar{S})$. As S is comprised of sources and intermediates, clearly,

$$val\ f = f_{out}(X) - f_{in}(X) = f_{out}(S) - f_{in}(S)$$

since the intermediates contribute nothing to flow value.

Min-Cut Max-flow Theorem...

- Consider an arc with both end points in S: its flow is counted in both $f_{out}(S)$ and $f_{in}(S)$, and thus makes no net impact on the flow value. Therefore, the only arcs flows which positively impact *val f* are those originating in S and terminating in $\bar{S}$, which are precisely flows over cut K. Thus,

$$val\ f \leq \sum_{a \in K} f(a) \leq \sum_{a \in K} c(a) = cap\ K.$$

  Some applications:

- Given any digraph with at least two vertices, designate some vertex x the source and vertext y the sink, and let all arcs have unit capacity.

Some applications

- Then a flow on this network counts (via its value) a number of arc-disjoint directed x, y-paths, and a cut counts (via its capacity) a number of arcs whose deletion destroyes all x, y-paths.
- Menger's Theorem: Let x, y be distinct vertices of a digraph D. The maximum number of arc-disjoint directed x, y-paths in D equals the minimum number of arcs whose deletion destroyes all diretced x, y-paths in D.
- Similarly, let x, y be distict vertices of a graph G. The maximum number of edge-disjoint x, y-paths in G equals the minimum number of edges whose deletion destroyes all x, y-paths in G.

Finding minimum spanning trees

- A minimum spanning tree of an undirected graph can be easily obtained using classical algorithms by *Kruskal* or *Prim*. A number of algorithms have been proposed to enumerate all spanning trees of an undirected graph.
- Let in a undirected graph $G = (V, E)$, $E = \{(u, v) \mid u, v \in V\}$. In weighted graph, $w : E \rightarrow \mathbb{R}$, which assigned weight each edge (called cost).
- Spanning tree is graph consisting all the vertices, and all are connected my minimum number of edges.

Finding minimum spanning trees

- **Kruskal's algorithm.** Repeat until entire new graph M has $n-1$ edges, and initially M was empty. Add to M the shortest edge, which does not make it a circle.
- **Prim's Algorithm.** Repeat follwoing until M has $n-1$ edges, with M initially empty. Add the shortest edge with $v_i \in M$ and $v_j \notin M$.

Finding all the spanning trees

- *Cayley's* formula counts the number of spanning trees on a complete graph. Cayley's formula is a result in graph theory named after Arthur Cayley. It states that for every n, the number of trees on n labeled vertices is n_{n-2}. There are $2^{2-2} = 1$ trees in K_2, $3^{3-2} = 3$ trees in K_3, and $4^{4-2} = 16$ trees in K_4.
- Suppose we have V nodes and E edges.
  1. Get all edges of the graph
  2. Get all possible combinations of $V - 1$ out of E edges.
  3. Filter out non-spanning-tree out of the combinations (for a spanning tree, all nodes inside one set of $V - 1$ edges should appear exactly once)

  (Alternatively, make all edges of equal weight, and then find all minimum spanning trees).