Ackermann’s Function

Presentation by
Upasana Pujari
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Function Definition

Ackermann's function was defined in 1920s by German mathematician and logician Wilhelm Ackermann (1896-1962).

A(m,n),   m,n $\in$ **N** such that**,**

$$A(0, n) = n + 1, \quad n \geq 0;$$

$$A(m,0) = A(m-1, 1), \quad m > 0;$$

$$A(m,n) = A(m-1, A(m, n-1)), \quad m, n > 0;$$

Example - 1

$$
\begin{aligned}
A(1, 2) &= A(0, A(1, 1)) \\
&= A(0, A(0, A(1, 0))) \\
&= A(0, A(0, A(0, 1))) \\
&= A(0, A(0, 2)) \\
&= A(0, 3) \\
&= 4
\end{aligned}
$$

Simple addition and subtraction!!

Example - 2

A (2, 2) = A (1, A (2, 1))
= A (1, A (1, A (2, 0)))
= A (1, A (1, A (1, 1)))
= A (1, A (1, A (0, A (1, 0))))
= A (1, A (1, A (0, A (0, 1))))
= A (1, A (1, A (0, 2)))
= A (1, A (1, 3))
= A (1, A (0, A (1, 2)))
= A (1, A (0, A (0, A (1, 1))))
= A (1, A (0, A (0, A (0, A (1, 0)))))
= A (1, A (0, A (0, A (0, A (0, 1)))))
= A (1, A (0, A (0, A (0, 2))))
= A (1, A (0, A (0, 3)))
= A (1, A (0, 4))

= A (1, 5)
= A (0, A (1, 4))
= A (0, A (0, A (1, 3)))
= A (0, A (0, A (0, A (1, 2))))
= A (0, A (0, A (0, A (0, A (1, 1)))))
= A (0, A(0, A(0, A(0, A(0, A(1, 0))))))
= A (0, A(0, A(0, A(0, A(0, A(0, 1))))))
= A (0, A (0, A (0, A (0, A (0, 2)))))
= A (0, A (0, A (0, A (0, 3))))
= A (0, A (0, A (0, 4)))
= A (0, A (0, 5))
= A (0, 6)
= 7

Ackermann's Function

- It is a well defined total function.
- Computable but not primitive recursive.
- Grows faster than any primitive recursive function.
- It is μ-recursive.

A(m,n)	n = 0	n = 1	n = 2	n = 3	n = 4
m = 0	1	2	3	4	5
m = 1	2	3	4	5	6
m = 2	3	5	7	9	11
m = 3	5	13	29	61	125
m = 4	13	65533	$2^{65533} - 3$	A(3, $2^{65533} – 3$)	A(3, A(4,3))
m = 5	65533	A(4, 65533)	A(4, A(5,1))	A(4, A(5,2))	A(4, A(5,3))
m = 6	A(4,65533)	A(5, A(5,1))	A(5, A(6,1)	A(5, A(6,2)	A(5, A(6,3)

Equivalent Definition

$A(0, n) = n + 1$

$A(1, n) = 2 + (n + 3) - 3$

$A(2, n) = 2 \times (n + 3) - 3$

$A(3, n) = 2^{n + 3} - 3$

$A(4, n) = \underbrace{2^{2^{2 \cdots 2}}}_{(n + 3 \text{ terms})} - 3$

…

Terms of the form $2^{2^{2 \cdots 2}}$ are known as power towers.

Arrow Notation

- Invented by Knuth (1976)
- Used to represent large numbers such as the power towers and Ackermann numbers.

$$m \uparrow n \quad = m^n$$

$$m \uparrow\uparrow n \quad = \underbrace{m \uparrow \ldots \uparrow m}_{n} = \underbrace{m^{m^{\cdots^{m}}}}_{n}$$

$$m \uparrow\uparrow\uparrow n = \underbrace{m \uparrow\uparrow \ldots \uparrow\uparrow m}_{n} \uparrow\uparrow m = m \uparrow\uparrow \ldots \uparrow\uparrow \underbrace{m^{m^{\cdots^{m}}}}_{m}$$

Sample Implementation

- Recursive version

```
function ack (m, n)
   if m = 0
      return n+1
   else if m > 0 and n = 0
      return ack (m-1, 1)
   else if m > 0 and n > 0
      return ack (m-1, ack (m, n-1))
```

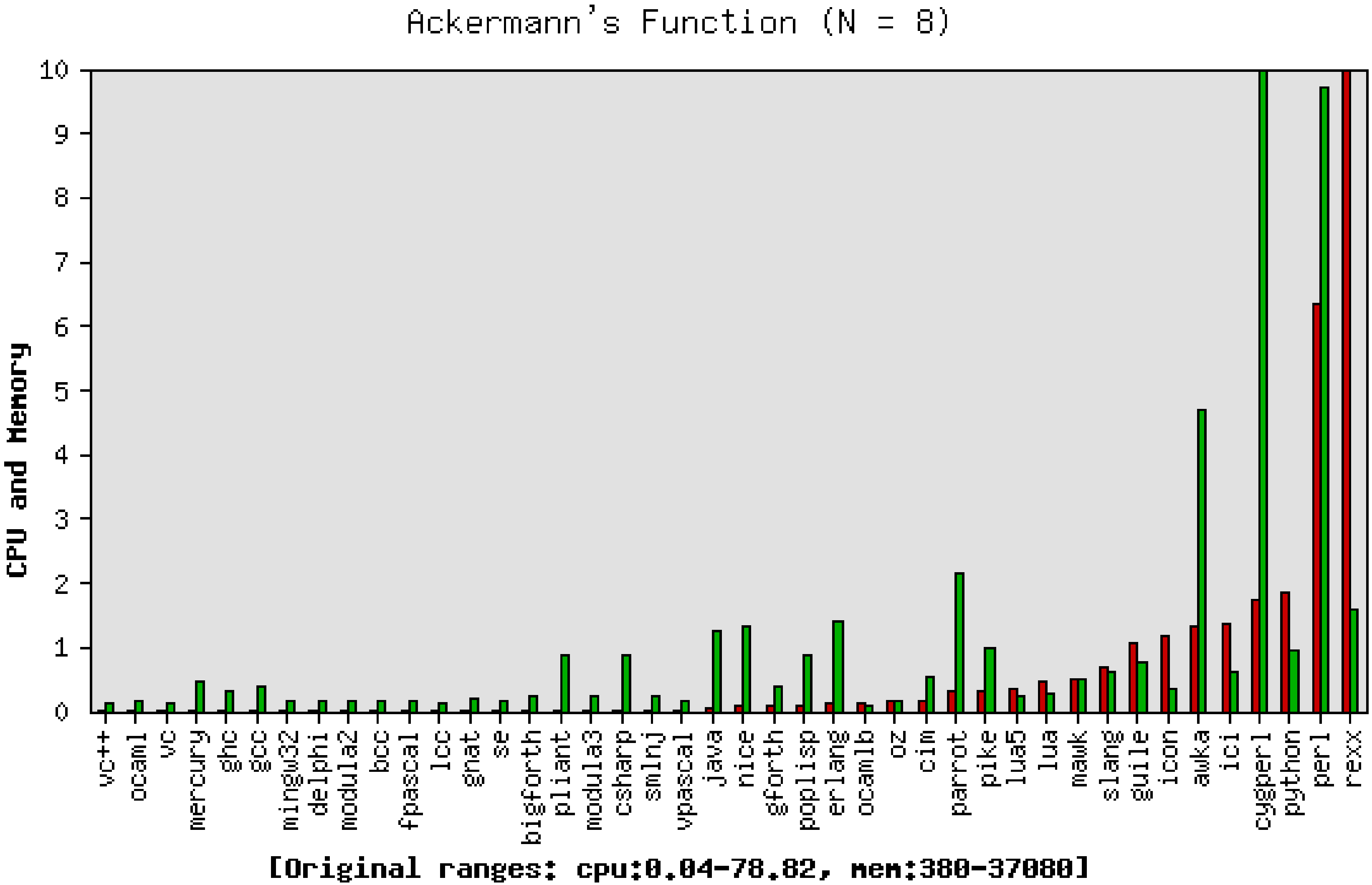
- Partially iterative version

```
function ack (m, n)
   while m ≠ 0
      if n = 0
         n := 1
      else
         n:= ack (m, n-1)
      m := m - 1
   return n + 1
```

Applications

- In Computational complexity of some algorithms
  - Union-find algorithm
  - Chazelle’s algorithm for minimum spanning tree
- In theory of recursive functions
- As a benchmark of a compiler’s ability to optimize recursion
- In specifying huge dimensions in certain theories such as Ramsey Theory

Benchmarking



Questions?