Axiomatic systems

KR Chowdhary
Professor & Head
Email: kr.chowdhary@gmail.com

Department of Computer Science and Engineering
MBM Engineering College, Jodhpur

March 19, 2013

Important Terms

- **Undefined and Defined Terms:** The list of undefined terms are the technical words that use in this subject. It may be noted that Euclid had tried to define every term but now we know that is an impossible task so we simply take certain keywords to be undefined and work from there.
- **Examples:** Keywords such as point and line are undefined terms in geometry.
- The aim is to start with a minimal number of undefined terms and the define other technical terms using the undefined terms and previously defined terms.

Important Terms

- **Axioms:** The second part of an axiomatic system is a list of axioms. Axioms (or postulates) are statements that are accepted without proof. They are where the subject begins and everything else in the system is logically deduced from them.
- In a way the axioms act as definitions for undefined terms, for example points and lines are undefined terms but in axioms we say what it is about points and lines that will be used in our development of geometry.
- **Theorems:** The last part of an axiomatic system consists of theorems and their proofs. The words theorem and proposition are used synonymously. In this part of an axiomatic system we work out the logical consequences of the axioms.

Models of Axiomatic Systems

- **Models:** A model for an axiomatic system is a realization of the axioms in some mathematical setting in which the universe is specified, all undefined terms are interpreted and all axioms are true.
- **Example:**Axioms of set theory: $\wedge, \vee, \neg, \rightarrow$ along with propositional variables.
- **Axiomatic System:** Set of axioms and inference rules.

Models of Axiomatic Systems

- **A Perfect Axiomatic System:** derives all true statements (in the domain), and no false statements, starting from a finite number of axioms and following mechanical inference rules.
- What is an *incomplete* axiomatic system?
- What is an *inconsistent* axiomatic system?
- Why is an *incomplete* axiomatic system preferable to an *inconsistent* one?
- What is an *independent* axiomatic system ?

Independence and Consistency in Axiomatic Systems

- An axiomatic system is consistent if it is free of contradictions i.e. the ability to derive both a statement and its negation from the system's axioms.
- Every axiom is independent of the others, none of these is a logical consequence of the others or in other words, it is not a theorem that can be derived from other axioms in the system. A system will be called independent if each of its underlying axioms is independent.
- Although independence is not a necessary requirement for a system, consistency is.

Models and Consistency of Axiomatic Systems

- Models play an important role in deciding these issues. Thus, a model for a set of axioms in some concrete system always establishes (at least) relative consistency.
- Consistency is often difficult to prove. One method for showing that an axiomatic system is consistent is to use a model. When a concrete model has been exhibited, we say we have established the *absolute consistency* of the axiomatic system. If we exhibit an abstract model where the axioms of the first system are theorems of the second system, then we say the first axiomatic system is *relatively consistent*. Relative consistency is usually the best we can hope for since concrete models are often difficult or impossible to set up.

Models and Independence of Axiomatic Systems

- If a model can be constructed for all but one of the axioms in a certain axiomatic system and if the one excluded axiom is false in that model then the excluded axiom cannot be proven from the rest, then the axiom is said to be independent. If this is true for each axiom then the system as a whole is *independent*.

An Important Example of an Axiomatic System

- *Incidence Geometry:* Undefined terms are point, line and lie on (as in "point P lies on line l").
- **Axioms for points and lines:**
- Axiom (I-1): For every pair of distinct points P and Q there exists exactly one line l such that both P and Q lie on l.
- Axiom (I-2): For every line l there exists at least two distinct points P and Q such that both P and Q lie on l.
- Axiom (I-3): There exists three points that do not all lie on any one line.

- Let Γ is set of formulas, and λ is a formula.

  If $\Gamma \models \alpha \Rightarrow \Gamma \vdash \alpha$, then the system is *sound*.

  If $\Gamma \vdash \alpha \Rightarrow \Gamma \models \alpha$, then the system is *complete*.

  where, $\models$, and $\vdash$ are signs for "logically follows" and "deductions", respectively