Turing Machine Extensions, and Enumerators

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Ways to Extend Turing Machines

Many variations have been proposed:

- Multiple tape Turing machine
- Multiple track Turing machine
- Two dimensional Turing machine
- Multidimensional Turing machine
- Two-way infinite tape
- Non-determinic Turing machine
- Combinations of the above
- Theorem: The operations of a TM allowing some or all the above extensions can be simulated by a standard TM.The extensions do not give us machines more powerful than the TM.
- The extensions are helpful in designing machines to solve particular problems.

Multiple Tape TM

Variants of TM:

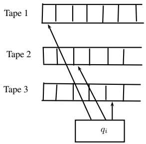
- For example, in two tape Turing machine, each tape has its own read-write head, but the state is common for all tapes and all heads.
- In each step (transitions) TM reads symbols scanned by all heads, depending on head position and current state, each head writes, moves R or L, and control-unit enter into new state.
- Actions of heads are independent of each other.
- Tape position in two tapes: $[x, y]$, x in first tape, and y in second, and δ is given by:

  $\delta(q_i, [x, y]) = (q_j, [z, w], d, d), \quad d \in \{L, R\}$

δ	a, a	B, a	a, B	B, B
q_0	q_1, a, b, L, R	q_2, b, B, L, L	$q_0, b, B, L, R,$	$\ldots$

- A standard TM is multi-tape TM with single tape.
- Example: These multi-tape TMs are better suited for specific applications, e.g., copying string from one tape to another tape. However, their time-complexity remains **P** only.

Multiple Tape TM



A transition in a multi-tape Turing machine, for $k \geq 1$ number of tapes:

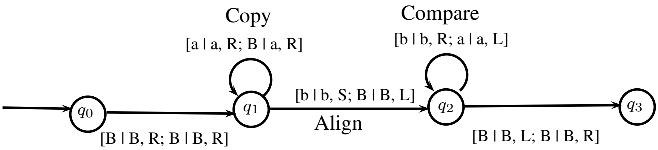
$\delta : (Q - H) \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R\}^k$

$\delta(q_i, a_1, \ldots, a_k) = (q_j, (b_1, \ldots, b_k), (d_1, \ldots, d_k))$

The steps to carry out a transition are:

1. change to next state;
2. write one symbols on each tape;
3. independently reposition each tape heads.

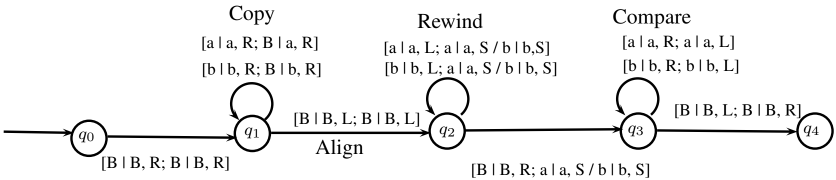
Two-tape TM to recognize language $L = \{a^n b^n \mid n \geq 0\}$



Working: with The original string w is on tape 1

1. Copies the string a's on tape 2.
2. Moves head 1 to begin of b's, and head on last a's
3. Compare number of b's tape 1 with number of b's on tape 2 by moving heads in opposite directions.
4. Time Complexity: $1+n+1+n+1 = O(n+3) = O(n) =$ Polynomial (**P**) (Compare it with standard TM for same problem with complexity $O(n^2)$)

Two-tape TM to recognize language $L = \{ww^R \mid w \in \{a,b\}^*\}$

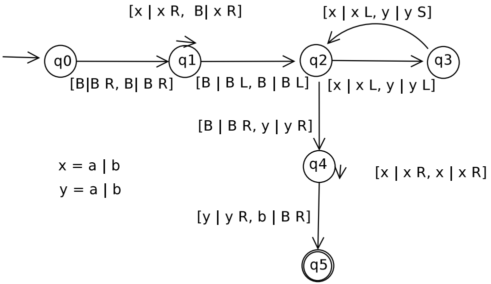


Working: with The original string w is on tape 1

1. Copies the string w on tape 2.
2. Moves head 1 to extreme left (rewind)
3. Aligns both heads on opposite, i.e, head 1 to extreme left, head 2 to extreme right
4. Compare by moving heads in opposite directions.
5. Time Complexity: $1+n+1+n+1+n+1 = O(3n+4) = O(n) =$ Polynomial (**P**)

Multi-Tape TM

Example: Construct 2-tape TM to recognize the language $L = \{ww | w \in \{a, b\}^*\}$.



steps:(Note: $x \in \{a, b\}$, $y \in \{a, b\}$)

1. Initially the string *ww* is on tape-1. Copy it to tape-2, at the end both R/W heads are at right most.
2. *Move both heads left:* Head-1, 2-steps and head-2, 1-step each time.
3. *Move both heads right*, each one step. Head-1 moves in first w of ww and head-2 moves in second w, comparing these w in q_4. In $q_3 \rightarrow q_4$ transition, the move '$y|y\ S$' keeps head2 *stationary*.

Multiple Tape TM

Simulation of a three-tape TM M by standard TM S:

- Let the contents of a three tape TM M are:

  $0\mathbf{1}010B$ Tape 1: bold character is R/W head position

  $aa\mathbf{a}B$ Tape 2: bold character is R/W head position

  $\mathbf{b}aB$ Tape 3: bold character is R/W head position

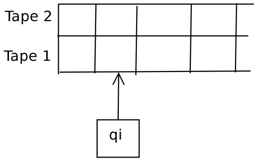
- Tape contents on simulated standard TM S:

  $0\mathbf{1}010\#aa\mathbf{a}\#\mathbf{b}aB$, # is separator for contents of 3 tapes.

- In practice, S leaves an extra blank before each symbol to record position of read-write heads
- S reads the symbols under the virtual heads (L to R).
- Then S makes a second pass to update the tapes according to the way the transition function of M dictates.
- If, at any point S moves one of the virtual heads to the right of #, it implies that head moved to unread blank portion of that tape. So S writes a blank symbol in the right most of that tape. Then continues to simulate.

  $\Rightarrow$ control will need a lot more states.

Multiple track TM



- The tape is divided into tracks. A tape position in n-track tape contains n symbols from tape alphabets.
- Tape position in two-track is represented by $[x, y]$, where x is symbol in track 1 and y is in tack-2. The states, Σ, Γ, q_0, F of a two-track machine are same as for standard machine.
- A transition of a two-track machine reads and writes the entire position on both the tracks.
- δ is: $\delta(q_i, [x, y]) = [q_j, [z, w], d]$, where $d \in \{L, R\}$. The input for two-track is put at track-1, and all positions on track-2 is initially blank. The acceptance in multi-track is by final state.
- Languages accepted by two-track machines are Recursively Enumerable languages.

Multi-track Turing Machine = Standard TM

Theorem

A language is accepted by a two-track TM M if and only if it is accepted by a standard TM M'.

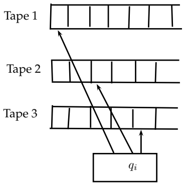
Proof.

- *Part 1:If L is accepted by standard TM M' then it is accepted by two-track TM M also (for this, simply ignore 2nd track). Hence track content is tuple $[a, B]$.*

  Part 2:
- *Let $M = (Q, \Sigma, \Gamma, \delta, q_0, H)$ be two track machine. Find one equivalent standard TM M'*
- *Create ordered pair $[x, y]$ on single tape machine M'.*
- *$M' = (Q, \Sigma \times \{B\}, \Gamma \times \Gamma, \delta', q_0, F)$ with*
  δ' as $\delta'(q_i, [x, y]) = \delta(q_i, [x, y])$.

$\square$

Construct a 3-tape TM to recognize the set $\{a^k | k \text{ is a perfect square}\}$



1. Tape 1 holds the input, tape 2 holds progressive k^2 and tape 3 the progressive k
2. keep $T_2 = T_3 = \varepsilon$.
3. If Input is compared with T_2, by scanning both. If both reach to B together, accept, and terminate. Else, put a at T_2 and T_3.
4. If $T_2 \equiv T_1$, accept and terminate, else if $|T_2| > |T_1|$, reject input, else $(T_2 < T_1)$,

   append T_2 with $2 \times T_3$ and append a to T_2 (this changes content of T_2 from a^{k^2} to a^{k^2+2k+1}.
5. append a to T_3 (increment t_3).
6. go back to step 3.

Two-way infinite tape

- There is single tape M which extends from $-\infty$ *to* $+\infty$. One R-W head, $M = (Q, \Sigma, \delta, q_0, F)$

  . . . -3 -2 -1 0 1 2 3 . . . , is square sequence on TM, with R-W head at 0

  This can be simulated by a two-tape TM:

- $M' = (Q' \cup \{q_s, q_t\}) \times \{U, D\}, \Sigma', \Gamma', f')$, where U = up tape head, D = down tape head, $\Sigma' = \Sigma, \Gamma' = \Gamma \cup \{B\}$, and

  $F^{'} = \{[q_i, U], [q_i, D] | q_i \in F\}$. Initial state of $M^{'}$ is pair $[q_s, D]$. A transition from this writes B in U tape at left most position. Transition from $[q_t, D]$ returns the tape head to its original position to begin simulation of M.

Multi-Dimensional Tape:

- Single R-W head, but multiple tapes exists. Let the Dimensions be 2D. For each input symbol and state, this writes a symbols at current head position, moves to a new state, and R-W head moves to left or right or up or down.

  Simulate it on 2-tape TM:

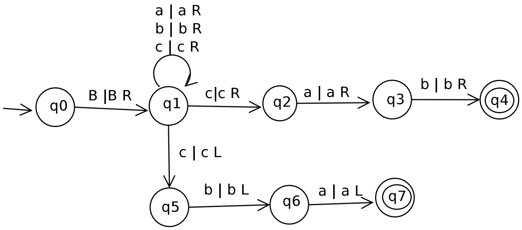
- copy each row of 2-D tape on 2nd tape of 2-tape TM. When 2D TM moves head L or R, move the head on 2nd-tape of two-tape also L or R. When 2D head moves up, 2nd tape of two-tape scans left until it finds $*$. As it scans, it writes the symbols on tape-1. Then scans and puts remaining symbols on tape-1. Now it simulates this row (on tape-1).

Nondeterministic TM (NDTM)

- NDTM has finite number of choices of moves; components are same as standard TM; may have > 1 move with same input and state pair, $(Q \times \Sigma)$. Nondeterminism is like FA and PDA.
- A NDTM machine accepts by halting if there is at least one computation sequence that halts normally when run with input w.
- Example: Find if a graph of n nodes has a connected subgraph of k nodes. For this no efficient algorithm exists. A Non-exhaustive based solution is Guess and check.
  1. NDTM: Arbitrarily choose a move when more than one possibility exists for $\delta(q_i, a)$.
  2. Accept the input if there is at least one computation sequence that leads to accepting state (however, the converse is irrelevant).
- To find a NDTM for ww input, $w \in \Sigma^*$, you need to guess the mid point. A *NDTM* may specify any number of transitions for a given configuration, i.e. $\delta : (Q - H) \times \Gamma \rightarrow subset \;\; of \;\; Q \times \Gamma \times \{L, R\}$

Construct a NDTM to accept *ab* preceded or followed with *c*

Example: $w = ucv$, where c is preceded by or followed by ab, and $u, v \in \{a, b, c\}^*$



Approach: Read input a, b, c and write a, b, c respectively, and move to R in each, at start state. Then with input c, Nondeterministically decide c, a, b by moving R in three states transitions or decide c, b, a by moving L in three other states transitions (i.e., abc)

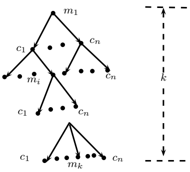
Simulation of NDTM on Standard TM

- To accomplish the transformation of a NDTM into a deterministic TM, we show that multiple computations of a single input string can be sequentially generated and examined.
- A NDTM produces multiple computations for a single string. We show that multiple computations $m_1, \dots, m_i, \dots, m_k$ for a single input string can be sequentially generated and applied.(A computation m_i is $\delta(q_i, a) = [q_j, b, d]$, where $d \in \{L, R\}$.
- These computations can be systematically produced by adding the alternative transitions for each $Q \times \Sigma$ pair. Each m_i has $1 : n$ number of transitions. If $\delta(q_i, x) = \phi$, the TM halts.
- Using the ability to sequentially produce the computations, a *NDTM* M can be simulated by a 3-tape deterministic TM M'.
- Every nondeterministic TM has an equivalent 3-tape Turing machine, which, in turn, has an equivalent standard TM (1-tape Turing machine).

Simulation of a NDTM M by 3-tape TM M'

Simulation of NDTM by 3-tape DTM:

- Approach: A NDTM may have more than one transition for same input (state $\times$ input symbol) pair. We may call these configurations as $c_1 ... c_n$.



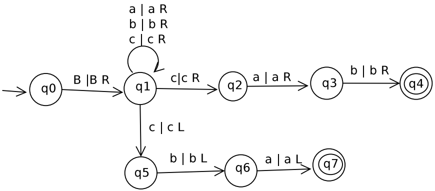
- If the computation sequence $m_1, \ldots, m_i, \ldots, m_k$, leads to solution, then for a NDTM these are the steps for solution, where $m_i \in \{c_1, \ldots, c_n\}$. For exhaustive solution, complexity is decided by tree height k, hence complexity is k^n.
- To perform the simulation of M on M', all the computation sequence, for each (state, input) pair are systematically generated.
- The tree created can be searched in DFS or BFS, until accepting/halting state is reached.
- However, DFS is not preferred because, it may go infinity. Therefore, the generated states are searched in BFS.

Simulation of a NDTM M by 3-tape TM m'

- Tape-1 of M' stores the input string, tape-2 simulates the tape of M, and tape-3 holds sequence $m_1, \ldots, m_i, \ldots, m_k$ to guide the simulation.
- Computation of $M^{'}$ consists following:
  1. A sequence of inputs $m_1, \ldots, m_i, \ldots, m_k$, where each $i, (i = 1 : n)$ is written on tape-3. $(m_i \in \{c_1, \ldots, c_n\})$
  2. Input string is copied on tape-2.
  3. Computation of M defined by sequence on tape-3 is simulated on tape-2.
  4. If simulation halts prier to executing k transitions, computations of M' halts and accepts input, else
  5. the Next sequence is generated on tape-3 and computation continues on tape-2.

Simulation of a NDTM M by 3-tape TM m'

Let us consider the NDTM M given below to be simulated on 3-tape DTM M'. Let us assume that input string to NDTM is $w = acab$.



Since, there are maximum three transitions at any (state, input) pair, we take set $\{c_1, c_2, c_3\}$ as $\{1, 2, 3\}$. In case there is single transition only, we repeat that to make that count equal to 3. The following table shows all pairs of $(q \times \sigma)$, where $\sigma \in Sigma$.

$\delta(q_0, B)$	$(^1q_1, B, R)$	$\delta(q_2, a)$	$(^1q_3, a, R)$
	$(^2q_1, B, R)$		$(^2q_3, a, R)$
	$(^3q_1, B, R)$		$(^3q_3, a, R)$
$\delta(q_1, a)$	$(^1q_1, a, R)$	$\delta(q_1, c)$	$(^1q_1, c, R)$
	$(^2q_1, a, R)$		$(^2q_2, c, R)$
	$(^3q_1, a, R)$		$(^3q_5, c, R)$
$\delta(q_3, b)$	$(^1q_4, b, R)$	$\delta(q_5, b)$	$(^1q_6, b, R)$
	$(^2q_4, b, R)$		$(^2q_6, b, R)$
	$(^3q_4, b, R)$		$(^3q_6, b, R)$
...	...	...	...

Simulation of a NDTM M by 3-tape TM m'

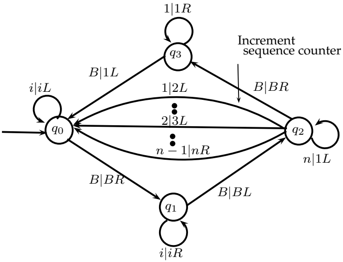
To simulate the NDTM on a 3-tape TM, following $m_1, \ldots, m_i, \ldots, m_k$ sequences are possible. All possible sequences are enumerated. The sequence numbers are given as per the table, given in the previous slide. Each sequence indicates a transition from one ID to next ID of the 3-tape DTM.

$q_0BacabB$ 1	$q_0BacabB$ 1	$q_0BacabB$ 2
$\vdash Bq_1acabB$ 1	$\vdash Bq_1acabB$ 1	$\vdash Bq_1acabB$ 2
$\vdash Baq_1cabB$ 1	$\vdash Baq_1cabB$ 2	$\vdash Baq_1cabB$ 3
$\vdash Bacq_1abB$ 1	$\vdash Bacq_2abB$ 1	$\vdash Bq_5acabB$
$\vdash Bacaq_1bB$ 1	$\vdash Bacaq_3bB$ 1	
$\vdash Bacabq_1B$	$\vdash Bacabq_4B$	
sequence=	sequence=	sequence
(1,1,1,1,1)	(1,1,2,1,1)	(2,2,3)

Therefore, what is required to generate the sequences as above: (1,1,1,1,1), then (1,1,2,1,1), then (2,2,3), in lexicographic order. We note, that the sequence (1,1,2,1,1) is accepting sequence; the moment such generated sequence of configurations is simulated on tape-2, the machine is halted.

Simulation of a NDTM: Generation of lexicographic sequences

Sequence generator is a TM, with no input, but output - sequence, whose generation is interleaved with the computation of the simulation of NDTM on tape-2 of M'. The figure below shows the sequence generation.



The simulation of NDTM M on a 3-tape DTM M' is as follows: (1) Input string w is put on tape 1, (2) w is copied on tape 2, (3) next sequence $(m_1..m_k)$ is generated on tape 3, (4) tape 2 is simulated for the moves (sequence) available on tape 3. (5) If any move leads to halting state, the machine is halted and w is accepted. (6) process is repeated from step 2. Note that sequence generation is non-terminating. Hence, if $w \notin L$, the process will continue indefinitely.

Turing Machine as language enumerator

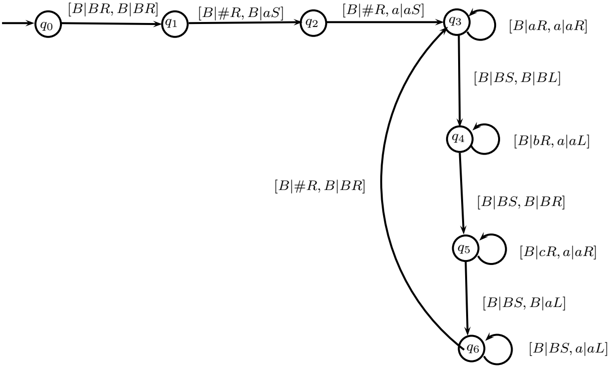
- TM can also be designed to enumerate a language. Such machines produce the exhaustive list of of string of the language, progressively longer and longer.
- Enumeration has no input, and its computation continues indefinitely if the language is infinite, as well as when it is finite.
- For enumeration, a TM of tape $k \geq 2$ is used, tape 1 is output tape, which would hold all the generated strings, separated by $\#$, and other tapes are working tapes.
- Output tape 1 has: $B\#u_1\#u_2\# \dots \#u_i\# \dots$, where $u_i \in L$.
- Tape head 1 always moves R, S, while others may move R, L, S.

Example

Enumerate all the strings for $L = \{a^n b^n c^n \mid n \geq 0\}$

Solution. The L can be generated by two-tape TM E with following steps: (1) Write $\#\#$ on tape 1 and 2 for ε. (2) add a on tape 2, (3) copy a equal to size of tape 2 on tape 1, then by same size the b is written on tape 1, then by same size c is written tape 1, followed with $\#$ (4) goto step 2.

The language $L = \{a^n b^n c^n \mid n \geq 0\}$ is suppose recognized by TM M. Let the enumerator E generates all the strings of L.



Turing Machine as language enumerator

Theorem

If L is enumerated by a TM then L is recursively enumerable.

Proof.

Let L is enumerated by a TM E of k tapes. We add a tape new tape $k+1$ to it. Let this new machine is M. Consider a string $w \in L$, and we write it on $k+1$th tape. Every time $\#$ is written on tape 1 by E, and its starts generating new string u_i, simultaneously it is compared with w on tape $k+1$, if found equal, M halts, otherwise the process of generating and compare is repeated for next string.
Since, when $w \in L$ machine M halts else continues indefinitely, L is recursively enumerable. $\square$

For enumeration it is necessary that lexicographic order (lo) of strings is generated. We can generate all the strings on alphabet $\Sigma = \{a_1, \ldots, a_n\}$ in lexicographic order using recursion as follows:

$$lo(\varepsilon) = 0,\ lo(a_i) = i,\ i = 1, n$$

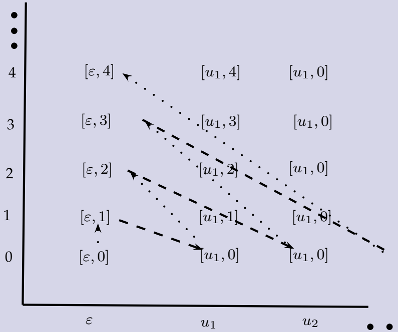
$$lo(a_i u) = i.n^{lenth(n)} + lo(u)$$

Theorem

For any alphabet Σ, there is TM E_{Σ^*} that enumerates Σ^* in lexicographic order.

Proof.

Let M be the TM that accepts L. The lexicographic ordering produces listing: ε, u_1, u_2, . . . , $\in \Sigma^*$. We Construct a table with columns as strings in Σ^* and rows as natural numbers, in the order as shown.



Turing Machine as language enumerator . . .

The $[i, j]$ entry in the table means "run the M for input u_i for j steps. The machine E is built to enumerate L such that enumeration of the ordered pairs are interleaved with the computation of M. The computation of E is a loop:

1. generate an ordered pair $[i, j]$
2. run a simulation of M with input u_i for j transitions or until the simulation halts.
3. If M accepts, write u_i on the output tape
4. continue with step 1.

If $u_i \in L$, the computation of M with input u_i halts and accepts after k transitions, for some number k.
Thus, u_i will be written on output tape of E when ordered pair $[i, j]$ is processed. The 2nd element of k ensures that simulation is terminated after k steps. Consequently, no non-nonterminating computation are allowed, and each string of Σ^* is examined.
This is one more proof of the theorem : "If a language is enumerated by TM then it is RE".