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## 19.1 TM as Language Recognizer

**Example 19.1** *Construct a TM that accepts the language  $L = \{a^n b^n \mid n \geq 0\}$ .*

Let  $M = (Q, \Sigma, \Gamma, \delta, s, H)$  be a Turing machine, which accepts the language  $L$ . Let us assume that the string  $w = a^n b^n$  exists on the tape, followed by infinite number of blank characters  $B$ . The working of  $M$  to recognize  $w$  is explained in following steps:

1.  $M$  replace left most  $a$  by  $X$ , and then head moves to right until it encounters left most  $b$ .
2. Replaces this  $b$  by  $Y$ , and then moves left to find the right most  $X$ . Then moves one step right to left most  $a$ .
3. Repeat Step 1 and 2 in order, i.e., 1, 2, 1, 2, ...
4. When searching for  $b$ , if  $M$  finds a blank character  $B$  (i.e.,  $|a^n| = |b^n|$ ) then  $M$  halts without accepting.
5. If  $a$  is not found but it finds  $b$  ( $a^n \neq b^n$ ), then  $M$  halts without accepting.
6. After changing  $b$  to  $Y$ , if  $M$  finds no more  $a$  then it checks that no more  $b$  remains. If this is true then  $a^n b^n$  is accepted by  $M$  (i.e.,  $|a^n| = |b^n|$ ).

Based on the above procedure,  $M$  can be specified as given below.

$$\begin{aligned} Q &= \{q_0, q_1, q_2, q_3, q_4\} \\ \Sigma &= \{a, b\} \\ \Gamma &= \{\triangleright, a, b, X, Y, B\} \\ s &= q_0 \\ H &= \{q_4\} \end{aligned}$$

The transition function for this machine is given below:

Fig. 19.1 shows a transition diagram for this TM accepting language  $L = \{a^n b^n \mid n \geq 0\}$ .

Table 19.1: Transition table for  $a^n b^n$  Language

state	a	b	X	Y	B
$q_0$	$(q_1, X, R)$			$(q_3, Y, R)$	$(q_4, B, L)$
$q_1$	$(q_1, a, R)$	$(q_2, Y, L)$		$(q_1, Y, R)$	
$q_2$	$(q_2, a, L)$		$(q_0, X, R)$	$(q_2, Y, L)$	
$q_3$				$(q_3, Y, R)$	$(q_4, B, L)$

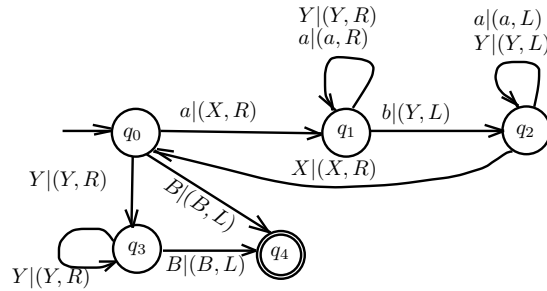


Figure 19.1: Transition diagram for TM accepting Language  $L = \{a^n b^n \mid n \geq 0\}$

Let  $w = aabb$ . The following are sequence of moves, which leads to the acceptance of  $w$ . A character  $B$  after  $aabb$  indicates end of the input string.

$$\begin{aligned}
 q_0 a a b b B &\vdash X q_1 a b b B \vdash X a q_1 b b B \vdash X a q_2 Y b B \vdash X q_2 a Y b B \\
 &\vdash q_2 X a Y b B \vdash X q_0 a Y b B \vdash X X q_1 Y b B \vdash X X Y q_1 b B \\
 &\vdash X X q_2 Y Y B \vdash X q_2 X Y Y B \vdash X X q_0 Y Y B \vdash X X Y q_3 Y B \\
 &\vdash X X Y Y q_3 B \vdash X X Y q_4 Y B
 \end{aligned}$$

The reader may verify that strings  $aab, abab, abb$  etc., are not accepted.

**Example 19.2** Finding complexity of TM for  $L = \{a^n b^n \mid n \geq 0\}$ .

The time (number of steps) taken by the 1-tape Turing machine for this can be calculated as follows:

- Assume that R/W head was initially pointing to input symbol  $a$  of  $a^n$ . In the first run, the head moves total  $n - 1$  steps in  $a$ 's and 1 step in  $b$  (total  $n$  steps) in forward direction, and equal in reverse, with total  $2n$  steps.
- This is repeated for  $n$ -times, making total of  $n * n = n^2$  steps.
- Next, there are total  $n$ -steps for transitions from  $q_0$  to  $q_3$ , plus for loop at  $q_3$  state. And finally one step for  $q_3$  to  $q_4$  transition.
- Thus total transitions are  $n^2 + n + 1$  for string  $a^n b^n$ . For an input  $|w| = n$ , this value is  $(n^2 + n + 1)/2$ . This gives a time of  $(n^2 + n + 1)/2$ , or time complexity of  $O(n^2)$ .

**Example 19.3** Show that the language  $L = \{a^n b^n c^n \mid n \geq 0\}$  is accepted by a TM.

We have just seen that language  $a^n b^n$  can be accepted by a TM. In that process  $a^n b^n$  gets converted to  $X^n Y^n$  before the TM halts.

To design a TM to accept  $a^n b^n c^n$ , we use the almost the previous technique and perform the following conversions:

1.  $aaabbbccc$  is converted to  $XaaYbbZcc$ ,
2. After the next operation, the tape contents are  $XXaYYbZZc$ ,
3. Finally, the tape contents are  $XXXYYYZZZ$ .

The above can be carried out by adding one more state after  $q_2$ , say  $q'_2$ . Hence, this proves that language  $a^n b^n c^n$  is accepted by a TM using the approach similar to method of example 19.1.