32002: AI (Substitution, Unifiers, and Disagreement sets.)	**Spring 2014**

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10.1 Substitutions and Unification

The following definitions are concerned with the operation of *instantiation*, i.e, substitutions of terms for variables in the well-formed expressions and in sets of well-formed expressions.

Substitution Components. A substitution component is any expression of the form T/V, where V is any variable and T is any term different from V. V is called variable of component T/V, and T is called term of the component.

Substitutions. A substitution is any finite set (possibly empty) of substitution components, none of the variables of which are same. If P is any set of terms, and the terms of the components of the substitution θ are all in P, we say that θ is a substitution over P. We write the substitution where components are $T_1/V_1, \ldots, T_k/V_k$ as $\{T_1/V_1, \ldots, T_k/V_k\}$, with the understanding that order of components is immaterial. We will use lowercase Greek letters θ, λ, μ denote substitutions.

Instantiations. If E is any function string of symbols, and $\theta = \{T_1/V_1, \ldots, T_k/V_k\}$ is any substitution, then the instantiation of E by θ is the operation of replacing each occurrence of variable $V_i, 1 \leq i \leq k$, in E by occurrence of the term T_i. The resulting string denoted by $E\theta$ is called an instance of E by θ, i.e., if E is the string $E_0 V_{i_1} E_1 \ldots V_{i_n} En$, then $E\theta$ is the string $E_0 T_{i_1} E_1 \ldots T_{i_n} En$. Here, none of the substrings E_j of E contain occurrences of variables $V_1, \ldots, V_k$, some of E_j are possibly *null*, n is possibly 0, and each V_{i_j} is an occurrence of one of the variables $V_1, \ldots, V_k$. Any string $E\theta$ is called an instance of the string E.

If C is any set of strings and θ is substitution, then the instance of C by θ is the set of all strings $E\theta$, where E is in C. We denote this set by $C\theta$, and say that it is an instance of C

Composition of Substitutions. If $\theta = \{T_1/V_1, \ldots, T_k/V_k\}$ and λ are any two substitutions, then the composition of θ and λ denoted by $\theta\lambda$ is union $\theta^{'} \cup \lambda^{'}$, defined as follows:

The $\theta^{'}$ is set of all components $T_i\lambda/V_i, 1 \leq i \leq k$, such that $T_i\lambda$ (λ substituted in θ) is different from V_i, and $\lambda^{'}$ is set of all components of λ whose variables are not among $V_1, \ldots, V_k$.

Example 10.1 *Find out the composition of* $\{x/y, w/z\}, \{v/x\}$, *and* $\{A/v,\ f(B)/w\}$.

Solution. Let us assume that $\theta = \{x/y, w/z\}$, $\lambda = \{v/x\}$ and $\mu = \{A/v, f(B)/w\}$. To find the composition $\lambda\mu$, A is is substituted for v, and v is then substituted for x. Thus, $\lambda\mu = \{A/x, f(B)/w\}$. When result of $\lambda\mu$ is substituted in θ, we get composition $\theta\lambda\mu = \{A/y, f(B)/z\}$. □

Example 10.2 *Find out the composition of* $\{g(x, y)/z\}$, *and* $\{A/x, B/y, C/w, D/z\}$.

Solution. $\{g(A, B)/z, A/x, B/y, C/w\}$. $\{D/z\}$ has not been included in the resultant substitution set, because otherwise, there will be two terms for the variable z, one $g(A, B)$ and other D. □

One of the important property of substitution is that, if E is any string, and $\sigma = \theta\lambda$, then $E\sigma = E\theta\lambda$. It is straight forward to verify that $\epsilon\theta = \theta\epsilon = \theta$ for any substitution θ. Also, composition enjoys the associative property $(\theta\lambda)\mu = \theta\lambda\mu)$, so we may omit the parentheses in writing multiple compositions of substitutions. The substitutions are not in general commutative; i.e., it is generally not the case that $\theta\lambda = \lambda\theta$.

The point of the composition operation on substitution is that, when E is any string, and $\sigma = \theta\lambda$, the string $E\sigma$ is just the string $E\theta\lambda$, i.e., the instance of $E\theta$ by λ. The composition also have distributive property.

10.1.1 Unification

If A is any set of well-formed expressions and θ is a substitution, then θ is said to unify A, or to be a unifier of A, if $A\theta$ is singleton. Any set of well-formed expressions which has a unifier is said to be unifiable.

For Unification, a variable can be replaced by any term, including other variables and function expressions of arbitrary complexity, even including expressions that themselves contain variables. For example, the function expression, *father(jack)*, may be substituted for x in $human(x)$ to get *human(father(jack))*. Following are some valid substitution instances of $foo(x, a, zoo(y))$:

1. $foo(fred, a, zoo(z))$, where $fred$ is substituted for x and z for y, i.e., $\lambda_1 = \{fred/x, z/y\}$. Thus,
$$foo(fred, a, zoo(z)) = foo(x, a, zoo(y))\lambda_1.$$
2. $foo(w, a, zoo(jack))$, where $\lambda_2 = \{w/x, jack/y\}$; hence
$$foo(w, a, zoo(jack)) = foo(x, a, zoo(y))\lambda_2.$$
3. $foo(z, a, zoo(moo(z)))$, where $\lambda_3 = \{z/x, moo(z)/y\}$, hence;
$$foo(z, a, zoo(moo(z))) = foo(x, a, zoo(y))\lambda_3.$$

We use the notation x/y to indicate that x is substituted for the variable y. We also refer to these as bindings, so y is bound to x. A variable cannot be unified with a term containing that variable. So $p(x)$ cannot be substituted for x, because this would create an infinite regression: $p(p(p(...x)...)$.

If S is any unifier of a set of expressions E, and G is the most general unifier (i. e., simplest one) of that set of expressions, then for S applied to E there exists another unifier S' such that $ES = EGS'$ (where GS' is the composition of unifiers). In simple words, a most general unifier *mgu* is just what its name implies: the most general set of substitutions that can be found for the two expressions.

Example 10.3 *Given a unifier, obtain a more general unifier.*

Suppose you have two expressions $p(x)$ an $p(y)$. One way to unify these is to substitute any constant expression for x and y: $\{fred/x,\ fred/y\}$. But this is not the most general unifier, because if we substitute any variable for x and y, we get a more general unifier: $\{z/x, z/y\}$. The first unifier is a valid unifier, but it would lessen the generality of inferences that we might want to make.

Let, $E = \{p(x), p(y)\}$
Let, $S = \{fred/x, fred/y\}$
Let, $G = \{z/x, z/y\}$
Now, let $S^{'} = \{fred/z\}$
Then $ES = \{p(fred), p(fred)\}$
and $GS^{'} = \{fred/x, fred/y\}$
and therefore $EGS^{'} = \{p(fred), p(fred)\} = ES$. $\square$

So, given a unifier, you can always create a more general unifier. When both of these unifiers are composed and instantiate the expression E, you get the same instance as it was obtained with the earlier unifier.

References

[1] Chowdhary K.R. (2020) Logic and Reasoning Patterns. In: Fundamentals of Artificial Intelligence. Springer, New Delhi. `https://doi.org/10.1007/978-81-322-3972-7_3`