Hilbert and the Axiomatic Approach: Its Background and Development

Leo Corry, Tel-Aviv University
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Abstract

In this talk I will present a broad picture of the axiomatic approach that David Hilbert developed and promoted during a significant part of his career, as well as its wide-ranging scope of influence.

Hilbert's mathematical and scientific horizon was very broad and thoroughgoing. The axiomatic approach was one of the unifying themes of his overall activity, but it was only a part of a much more complex conception of what mathematics and science is all about. For many interesting historical reasons the essence of his approach came to be misunderstood. Eventually Hilbert became identified as the champion of formalism in mathematics, with the axiomatic method being the touchstone of this formalist conception. Recent historical work has helped us reach a more balanced and interesting picture.

In the talk, I will discuss the various threads that led to the consolidation of Hilbert's early conceptions. These threads are to be found in developments in late nineteenth-century geometry as well as in physics. Then, I will explain the ways in which Hilbert applied his views to geometry and physics in his own work, and how he thought they might be applied in other fields as well. I will also discuss the ways in which these views developed throughout the years. Finally, I will show how some of Hilbert's ideas were developed in various directions that essentially deviated from his original conceptions in various ways. The main examples of this appear in the USA school of Postulational Analysis in the early twentieth-century and in the work of the Bourbaki group.

In the discussion following the talk it will be possible to go into further details of any of these aspects, depending on the specific points of interest of the participants.

In 1899 the Göttingen mathematician David Hilbert (1862-1943) published his ground-breaking book *Grundlagen der Geometrie*. This book represented the culmination of a complex process that spanned the nineteenth century, whereby the most basic conceptions about the foundations, scope and structure of the discipline of geometry were totally reconceived and reformulated. Where Euclid had built the discipline more than two thousand years earlier on the basis of basic definitions and five postulates about the properties of shapes and figures in space, Hilbert came forward with a complex deductive structure based on five groups of axioms, namely, eight axioms of incidence, four of order, five of congruence, two of continuity and one of parallels. According to Hilbert's approach the basic concepts of geometry still comprise points, lines and planes, but, contrary to the Euclidean tradition, such concepts are never explicitly defined. Rather, they are implicitly defined by the axioms: points, lines and planes are any family of mathematical objects that satisfy the given axioms of geometry.

It is well known that Hilbert once explained his newly introduced approach by saying that in his system one might write "chairs", "tables" and "beer mugs", instead of "points", "lines" and "planes", and this would not affect the structure and the validity of the theory presented. Seen retrospectively, this explanation and the many times it was quoted were largely behind a widespread, fundamental misconception about the essence of Hilbert's approach to geometry. A second main reason for this confusion was that twenty years later Hilbert was the main promoter of a program intended to provide solid foundations to arithmetic based on purely "finitist" methods. The "formalist" program, as it became known, together with a retrospective reading of his work of 1900, gave rise to a view of Hilbert as the champion of a formalist approach to mathematics as a whole. This reading has sometimes been expressed in terms of a metaphor typically associated with Hilbert, namely, the "chess metaphor", which implies that 'mathematics is not about truths but about following correctly a set of stipulated rules'. For example, the leading French mathematician and founding Bourbaki member, Jean Dieudonné (1906-1992), who saw himself as a follower of what he thought was Hilbert's approach to mathematics said that, with Hilbert, "mathematics becomes a game, whose pieces are graphical signs that are distinguished from one another by their form".

This conception of Hilbert and his work is historically wrong. Hilbert's axiomatic approach was in no sense tantamount to axiomatic formalism. His approach to geometry was empiricist rather than formalist. I will bring here two quotations that summarize much of the essence of his conceptions and help give a more correct understanding of them. The first quotation is taken from a lecture delivered in 1919, where Hilbert clearly stated that:

> We are not speaking here of arbitrariness in any sense. Mathematics is not like a game whose tasks are determined by arbitrarily stipulated rules. Rather, it is a conceptual system possessing internal necessity that can only be so and by no means otherwise.

The second quotation is taken from a course taught in 1905 at Göttingen, where Hilbert presented systematically the way that his method should be applied to geometry, arithmetic and physics. He thus said:

> The edifice of science is not raised like a dwelling, in which the foundations are first firmly laid and only then one proceeds to construct and to enlarge the rooms. Science prefers to secure as soon as possible comfortable spaces to wander around and only subsequently, when signs appear here and there that the loose foundations are not able to sustain the expansion of the rooms, it sets about supporting and fortifying them. This is not a weakness, but rather the right and healthy path of development.

This latter quotation suggests that in Hilbert's view the axiomatic approach should never be taken as the starting point for the development of a mathematical or scientific theory. Likewise, Hilbert never saw axiomatics as a possible starting point to be used for didactical purposes. Rather, it should be applied only to existing, well-elaborated disciplines, as a useful tool for clarification purposes and for allowing its further development.

Hilbert applied his new axiomatic method to geometry in the first place not because geometry had some special status separating it from other mathematical enterprises, but only because its historical development had brought it to a stage in which fundamental logical and substantive issues were in need of clarification. As Hilbert explained very clearly, geometry had achieved a much more advanced stage of development than any other similar discipline. Thus, the edifice of geometry was well in place and as in Hilbert's metaphor quoted above, there were now some problems in the foundations that required fortification and the axiomatic method was the tool ideally suited to do so. Specifically, the logical interdependence of its basic axioms and theorems (especially in the case of projective geometry) appeared now as somewhat blurred and in need of clarification. This clarification, for Hilbert, consisted in defining an axiomatic system that lays at the basis of the theory and verifying that this system satisfied three main properties: independence, consistency, and completeness. Hilbert thought, moreover, that just as in geometry this kind of analysis should be applied to other fields of knowledge, and in particular to physical theories. When studying any system of axioms under his perspective, however, the focus of interest remained always on the disciplines themselves rather than on the axioms. The latter were just a means to improve our understanding of the former, and never a way to turn mathematics into a formally axiomatized game. In the case of geometry, the groups of axioms were selected in a way that reflected what Hilbert considered to be the basic manifestations of our intuition of space.

In 1900, moreover, "completeness" meant for Hilbert something very different to what the term came to signify after 1930, in the wake of the work of Gödel. All it meant at this point was that the known theorems of the discipline being investigated axiomatically would be derivable from the proposed system of axioms. Of course, Hilbert did not suggest any formal tool to verify this property. Consistency was naturally a main requirement, but Hilbert did not initially think that proofs of consistency would become a major mathematical task. Initially, the main question Hilbert intended to deal with in the *Grundlagen*, and elsewhere, was independence.

Indeed, he developed some technical tools specifically intended to prove the independence of axioms in a system, tools which became quite standard in decades to come. Still as we will see now, the significance and scope of these tools was transformed by some of those who used them, while following directions of research not originally envisaged or intended by Hilbert.

In the article attached to this text, I describe in a somewhat cursorily fashion the background to the consolidation of Hilbert's axiomatic approach. This background comprises developments in the late nineteenth century in both mathematics and physics. This background prepares the ground for a clear understanding of the common roots of Hilbert's scientific activities in fields that were traditionally considered as diverging aspects of his professional life. In particular it makes clear that the application of the axiomatic method to presenting the foundations of geometry and Hilbert's program for the axiomatization of physical theories were different sides of the same coin.

A detailed account of Hilbert's activities in physics and the role of the axiomatic method in these activities is beyond the scope of this presentation, and it appears in my book *Hilbert and the Axiomatization of Physics (1898-1918): From "Grundlagen der Geometrie" to "Grundlagen der Physik"* (Dordrecht: Kluwer - 2004). I would summarize three main points in the picture that arises from this account as follows:

1. Physics was a fundamental pillar of Hilbert's scientific worldview, and an organic component of his research and teaching activities *throughout* his career.
2. The axiomatization of physics was a fundamental connecting thread among all of his activities in physics, as well as the link to much of his work in pure mathematics.
3. Developments in physics influenced Hilbert's changing conceptions on the axiomatic approach, the role of axioms, and the relationship between mathematics (especially geometry) and physics.

In a similar vein, a detailed analysis of the way in which Hilbert's ideas were accepted by physicists and influenced developments in that field brings to a rather complex picture. Two revealing quotations illustrate at least a part of what comes up in this picture.

The first is a quotation by Max Born, taken from an article presented in 1922 on the occasion of Hilbert's 60th birthday. Max Born was among the prominent physicists of his generation, the one who was closer to Hilbert in all respects, who understood best the meaning of his axiomatic method, and who came close under its influence. He described the method in the following words:

> The physicist set outs to explore how things are in nature; experiment and theory are thus for him only a means to attain an aim. Conscious of the infinite complexities of the phenomena with which he is confronted in every experiment, he resists the idea of considering a theory as something definitive. He therefore abhors the word "Axiom", which in its usual usage evokes the idea of definitive truth. The physicist is thus acting in accordance with his healthy

> instinct, that dogmatism is the worst enemy of natural science. The mathematician, on the contrary, has no business with factual phenomena, but rather with logic interrelations. In Hilbert's language the axiomatic treatment of a discipline implies in no sense a definitive formulation of specific axioms as eternal truths, but rather the following methodological demand: specify the assumptions at the beginning of your deliberation, stop for a moment and investigate whether or not these assumptions are partly superfluous or contradict each other.

The second quotation is from Hermann Weyl, who was the most distinguished and prominent of Hilbert's more than sixty doctoral students. The relationship between the two was a rather complex one that spanned many levels. He was quite critical of Hilbert's approach to General Relativity as we know from letters between him and Einstein. Late in life he wrote about Hilbert and physics the following passage:

> The maze of experimental facts which the physicist has to take in account is too manifold, their expansion too fast, and their aspect and relative weight too changeable for the axiomatic method to find a firm enough foothold, except in the thoroughly consolidated parts of our physical knowledge. Men like Einstein and Niels Bohr grope their way in the dark toward their conceptions of general relativity or atomic structure by another type of experience and imagination than those of the mathematician, although no doubt mathematics is an essential ingredient. Thus, Hilbert's plans in physics never matured.

Nevertheless, the historical analysis of Hilbert's involvement with physical theories shows that he played a crucial role in shaping developments in many central threads at the beginning of the twentieth century, including the theory of relativity and quantum mechanics. As already stated, the axiomatic approach was a main unifying idea behind these activities.

On the origins of Hilbert's sixth problem: physics and the empiricist approach to axiomatization

Leo Corry

Abstract. The sixth of Hilbert's famous 1900 list of twenty-three problems is a programmatic call for the axiomatization of physical sciences. Contrary to a prevalent view this problem was naturally rooted at the core of Hilbert's conception of what axiomatization is all about. The axiomatic method embodied in his work on geometry at the turn of the twentieth-century originated in a preoccupation with foundational questions related with empirical science, including geometry and other physical disciplines at a similar level. From all the problems in the list, the sixth is the only one that continually engaged his efforts over a very long period, at least between 1894 and 1932.

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1. Introduction

Of the many important and brilliant plenary talks delivered in ICMs ever since the inception of this institution in 1897 in Zurich, none has so frequently been quoted and, possibly, none has had the kind of pervasive influence, as the one delivered by David Hilbert in 1900 at the second ICM in Paris, under the title of "Mathematical Problems". Rather than summarizing the state of the art in a central branch of mathematics, Hilbert attempted to "lift the veil" and peer into the development of mathematics in the century that was about to begin. He chose to present a list of twenty-three problems that in his opinion would and should occupy the efforts of mathematicians in the years to come. This famous list has been an object of mathematical and historical interest ever since.

The sixth problem of the list deals with the axiomatization of physics. It was suggested to Hilbert by his own recent research on the foundations of geometry. He proposed "to treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part." This problem differs from most others on Hilbert's list in essential ways, and its inclusion has been the object of noticeable reaction from mathematicians and historians who have discussed it throughout the years. Thus, in reports occasionally written about the current state of research on the twenty-three problems, the special status of the sixth problem is readily visible: not only has it been difficult to decide to what extent the problem was actually solved (or not), but one gets the impression that, of all the problems on the list, this one received

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the least attention from mathematicians throughout the century and that relatively little effort was directed at solving it ([11], [25]).

Many a historical account simply dismissed the sixth problem as a slip on Hilbert's side, as a curiosity, and as an artificial addition to what would otherwise appear as an organically conceived list, naturally connected to his broad range of mathematical interests (e.g., [26], p. 159). In fact, this is how Hilbert's interest in physical topics in general as well as his few, well-known incursions into physical problems have been traditionally seen. According to this view, these are seen as sporadic incursions into foreign territory, mainly for the purposes of finding some new applications to what would otherwise be purely mathematically motivated ideas. This is the case, for instance, with Hilbert's solution of the Boltzmann equation in kinetic theory of gases in 1912. Starting in 1902, most of Hilbert's mathematical energies had been focused on research related with the theory of linear integral equations, and his solution of the Boltzmann equation could thus be seen as no more than an application of the techniques developed as part of that theory to a particular situation, the physical background of which would be of no direct interest to Hilbert. An account in this spirit appears in Stephen G. Brush's authoritative book on the development of kinetic theory, according to which:

> When Hilbert decided to include a chapter on kinetic theory in his treatise on integral equations, it does not appear that he had any particular interest in the physical problems associated with gases. He did not try to make any detailed calculations of gas properties, and did not discuss the basic issues such as the nature of irreversibility and the validity of mechanical interpretations which had exercised the mathematician Ernst Zermelo in his debate with Boltzmann in 1896–97. A few years later, when Hilbert presented his views on the contemporary problems of physics, he did not even mention kinetic theory. We must therefore conclude that he was simply looking for another possible application of his mathematical theories, and when he had succeeded in finding and characterizing a special class of solutions (later called "normal") …his interest in the Boltzmann equation and in kinetic theory was exhausted. ([4], p. 448)

A further important physical context where Hilbert's appeared prominently concerns the formulation of the gravitational field-equations of the general theory of relativity (GTR). On November 20, 1915, Hilbert presented to the Royal Scientific Society in Göttingen his version of the equations, in the framework of what he saw as an axiomatically formulated foundation for the whole of physics. During that same month of November, Einstein had been struggling with the final stages of his own effort to formulate the generally covariant equations that lie at the heart of GTR. He presented three different versions at the weekly meetings of the Prussian Academy of Sciences in Berlin, before attaining his final version, on November 25, that is, five days *after* Hilbert had presented his own version.

Einstein had visited Göttingen in the summer of 1915 to lecture on his theory and on the difficulties currently encountered in his work. Hilbert was then in the audience and Einstein was greatly impressed by him. Earlier accounts of Hilbert's involvement with problems associated with GTR had in general traced it back to this visit of Einstein or, at the earliest, to the years immediately preceding it. As in the case of kinetic theory, this contribution of Hilbert was often seen as a more or less furtive incursion into physics, aimed at illustrating the power and the scope of validity of the "axiomatic method" and as a test of Hilbert's mathematical abilities while trying to "jump onto the bandwagon of success" of Einstein's theory.

In biographical accounts of Hilbert, his lively interest in physics has never been overlooked, to be sure, but it mostly has been presented as strictly circumscribed in time and scope. Thus for instance, in his obituary of Hilbert, Hermann Weyl ([24], p. 619) asserted that Hilbert's work comprised five separate, and clearly discernible main periods: (1) Theory of invariants (1885–1893); (2) Theory of algebraic number fields (1893–1898); (3) Foundations, (a) of geometry (1898–1902), (b) of mathematics in general (1922–1930); (4) Integral equations (1902–1912); (5) Physics (1910–1922). Weyl's account implies that the passage from any of these fields to the next was always clear-cut and irreversible, and a cursory examination of Hilbert's published works may confirm this impression. But as Weyl himself probably knew better than many, the list of Hilbert's publications provides only a partial, rather one-sided perspective of his intellectual horizons, and this is particularly the case when it comes to his activities related to physics.

Recent historical research has brought to light a very different picture of Hilbert's involvement with physics, and in particular of the real, truly central place of the ideas embodied in the sixth problem within the general edifice of Hilbert's scientific outlook. Hilbert's involvement with physical issues spanned most of his active scientific life, and the essence of his mathematical conceptions cannot be understood without reference to that involvement. More importantly, the famous "axiomatic approach" that came to be identified with Hilbert's mathematical achievements and with his pervasive influence on twentieth-century mathematics is totally misunderstood if it is not seen, in the first place, as connected with his physical interests. Under this perspective, the involvement with kinetic theory and GTR are seen as a natural outgrowth of the development of Hilbert's world of ideas, and by no means as sporadic, isolated incursions into unknown territories. Moreover, contrary to a commonly held view, the sixth problem is the only one in the entire list of 1900 that refers to an idea that continually engaged the active attention of Hilbert for a very long period of time, at least between 1894 and 1932 ([5]).

The key to a balanced understanding of the role of physics within Hilbert's intellectual horizon is found not so much in his publications, as it is in the complex academic network of personal interactions and diverse activities that he was continually part of. Especially worthy of attention is his teaching, first at Königsberg and – more importantly – after 1895 at Göttingen. At the mathematical institute established by Felix Klein, Hilbert became the leader of a unique scientific center that brought together a gallery of world-class researchers in mathematics and physics. One cannot exaggerate the significance of the influence exerted by Hilbert's thought and personality on all who came out of this institution. More often than not, these lectures were far from systematic and organized presentations of well-known results and established theories. Rather, Hilbert often used his lectures as a public stage where he could explore new ideas and think aloud about the issues that occupied his mind at any point in time. In a lecture held in commemorating his seventieth birthday, Hilbert vividly recalled how these lectures provided important occasions for the free exploration of yet untried ideas. He thus said:

> The closest conceivable connection between research and teaching became a decisive feature of my mathematical activity. The interchange of scientific ideas, the communication of what one found by himself and the elaboration of what one had heard, was from my early years at Königsberg a pivotal aspect of my scientific work. …In my lectures, and above all in the seminars, my guiding principle was not to present material in a standard and as smooth as possible way, just to help the student keep clean and ordered notebooks. Above all, I always tried to illuminate the problems and difficulties and to offer a bridge leading to currently open questions. It often happened that in the course of a semester the program of an advanced lecture was completely changed, because I wanted to discuss issues in which I was currently involved as a researcher and which had not yet by any means attained their definite formulation. ([16], p. 79)

The collection of Hilbert's lecture notes offers an invaluable source of information for anyone interested in understanding his scientific horizon and contributions.

2. Axiomatics and formalism

A main obstacle in historically understanding the significance of the sixth problem has been the widespread image of Hilbert as the champion of formalism in modern mathematics. The traditional association of Hilbert's name with the term "formalism" has often proved to be misleading, since the term can be understood in two completely different senses that are sometimes conflated. One sense refers to the so-called "Hilbert program" that occupied much of Hilbert's efforts from abut 1920. Although involving significant philosophical motivations, at the focus of this program stood a very specific, technical mathematical problem, namely, the attempt to prove the consistency of arithmetic with strictly finitist arguments. The point of view embodied in the program was eventually called the "formalist" approach to the foundations of mathematics, and it gained much resonance when it became a main contender in the so-called "foundational crisis" in mathematics early in the twentieth century.

Even though Hilbert himself did not use the term "formalism" in this context, associating his name with term conceived in this narrow sense seems to be essentially justified. It is misleading, however, to extend the term "Hilbert program" – and the concomitant idea of formalism – to refer to Hilbert's overall conception of the essence of mathematics. Indeed, a second meaning of the term formalism refers to a general attitude towards the practice of mathematics and the understanding of the essence of mathematical knowledge that gained widespread acceptance in the twentieth century, especially under the aegis of the Bourbaki group. Jean Dieudonné, for instance, explained what he saw as the essence of Hilbert's *mathematical* conceptions in a well-known text where he referred to the analogy with a game of chess. In the latter, he said, one does not speak about truths but rather about following correctly a set of stipulated rules. If we translate this into mathematics we obtain the putative, "formalist" conception often attributed to Hilbert ([6], p. 551): "mathematics becomes a *game*, whose pieces are graphical *signs* that are distinguished from one another by their form."

Understanding the historical roots and development of the sixth problem goes hand in hand with an understanding of Hilbert's overall conception of mathematics as being far removed from Dieudonné's chess-game metaphor. It also comprises a clear separation between the "Hilbert program" for the foundations of arithmetic, on the one hand, and Hilbert's lifetime research program for mathematics and physics and its variations throughout the years, on the other hand. In this regard, and even before one starts to look carefully at Hilbert's mathematical ideas and practice throughout his career, it is illustrative to look at a quotation from around 1919 – the time when Hilbert began to work out the finitist program for the foundations of arithmetic in collaboration with Paul Bernays – that expounds a view diametrically opposed to that attributed to him many years later by Dieudonné, and that is rather widespread even today. Thus Hilbert said:

> We are not speaking here of arbitrariness in any sense. Mathematics is not like a game whose tasks are determined by arbitrarily stipulated rules. Rather, it is a conceptual system possessing internal necessity that can only be so and by no means otherwise. ([16], p. 14)

The misleading conflation of the formalist aspect of the "Hilbert program" with Hilbert's overall views about mathematics and its relationship with physics is also closely related with a widespread, retrospective misreading of his early work on the foundations of geometry in purely formalist terms. However, the centrality attributed by Hilbert to the axiomatic method in mathematics and in science is strongly connected with thoroughgoing empiricist conceptions, that continually increased in strength as he went on to delve into ever new physical disciplines, and that reached a peek in 1915–17, the time of his most intense participation in research associated with GTR.

The axiomatic approach was for Hilbert, above all, a tool for retrospectively investigating the logical structure of *well-established and elaborated* scientific theories, and the possible difficulties encountered in their study, and never the starting point for the creation of new fields of enquiry. The role that Hilbert envisaged for the axiomatic analysis of theories is succinctly summarized in the following quotation taken from a course on the axiomatic method taught in 1905. Hilbert thus said:

> The edifice of science is not raised like a dwelling, in which the foundations are first firmly laid and only then one proceeds to construct and to enlarge the rooms. Science prefers to secure as soon as possible comfortable spaces to wander around and only subsequently, when signs appear here and there that the loose foundations are not able to sustain the expansion of the rooms, it sets about supporting and fortifying them. This is not a weakness, but rather the right and healthy path of development. ([5], p. 127)

3. Roots and early stages

Physics and mathematics were inextricably interconnected in Hilbert's scientific horizon ever since his early years as a young student in his native city of Königsberg, where he completed his doctorate in 1885 and continued to teach until 1895. Hilbert's dissertation and all of his early published work dealt with the theory of algebraic invariants. Subsequently he moved to the theory of algebraic number fields. But his student notebooks bear witness to a lively interest in, and a systematic study of, an astounding breadth of topics in both mathematics and physics. Particularly illuminating is a notebook that records his involvement as a student with the *Lehrbuch der Experimentalphysik* by Adolph Wüllner (1870). This was one of many textbooks at the time that systematically pursued the explicit reduction of all physical phenomena (particularly the theories of heat and light, magnetism and electricity) to mechanics, an approach that underlies all of Hilbert's early involvement with physics, and that he abandoned in favor of electrodynamical reductionism only after 1912.

In the intimate atmosphere of this small university, the student Hilbert participated in a weekly seminar organized under the initiative of Ferdinand Lindemann – who was also Hilbert's doctoral advisor – that was also attended by his good friends Adolf Hurwitz and Hermann Minkowski, by the two local physicist, Woldemar Voigt and Paul Volkmann, and by another fellow student Emil Wiechert, who would also become Hilbert's colleague in Göttingen and the world's leading geophysicist. The participants discussed recent research in all of branches of mathematics and physics, with special emphasis on hydrodynamics and electrodynamics, two topics of common interest for Hilbert and Minkowski throughout their careers. From very early on, fundamental methodological questions began to surface as part of Hilbert's involvement with both mathematics and physics.

On the mathematical side one may mention the intense research activity associated with the names of Cayley and Klein in projective geometry, concerning both the main body of results and the foundations of this discipline; the questions sparked by the discovery and publication of non-Euclidean geometries, which raised philosophical concerns to a larger extent than they elicited actual mathematical research; the introduction by Riemann of the manifold approach to the analysis of space and its elaboration by Lie and Helmholtz; the question of the arithmetization of the continuum as analyzed by Dedekind, which had also important foundational consequences for analysis; the gradual re-elaboration of axiomatic techniques and perspectives as a main approach to foundational questions in mathematics, especially in the hands of Grassmann and of the Italian geometers. Hilbert's intellectual debts to each of these traditions and to the mathematicians that partook in it – even though more complex and subtle than may appear on first sight – belong to the directly visible, received image of Hilbert the geometer.

What is remarkable, and virtually absent from the traditional historiography until relatively recently, is the extent to which similar parallel developments in physics played a fundamental role in shaping Hilbert's views on axiomatization. Very much like geometry, also physics underwent major changes throughout the nineteenth century. These changes affected the contents of the discipline, its methodology, its institutional setting, and its image in the eyes of its practitioners. They were accompanied by significant foundational debates that intensified considerably toward the end of the century, especially among German-speaking physicists. Part of these debates also translated into specific attempts to elucidate the role of basic laws or principles in physical theories, parallel in certain respects to that played by axioms in mathematical theories. As with geometry, foundational questions attracted relatively limited attention from practitioners of the discipline, but some leading figures were indeed involved in them.

From about 1850 on, physics became focused on quantification and the search for universal mathematical laws as its fundamental methodological principles, on the conservation of energy as a fundamental unifying principle, and very often on mechanical explanation of all physical phenomena as a preferred research direction. If explanations based on imponderable "fluids" had dominated so far, mechanical explanations based on the interaction of particles of ordinary matter now became much more frequent. In particular, the mechanical theory of ether gave additional impulse to the concept of "field" that would eventually require a mechanical explanation. Likewise, the kinetic theory of gases gave additional support to the foundational role of mechanics as a unifying, explanatory scheme. On the other hand, these very developments gave rise to many new questions that would eventually challenge the preferential status of mechanics and lead to the formulation of significant alternatives to it, especially in the form of the so-called "electromagnetic worldview", as well as in the "energicist" and the phenomenological approaches.

Beginning in the middle of the century, several physicists elaborated on the possibility of systematically clarifying foundational issues of this kind in physical theories, based on the use of "axioms", "postulates" or "principles". This was not, to be sure, a really central trend that engaged the leading physicists in lively discussions. Still, given the vivid interest on Volkmann in the topic, Hilbert became keenly aware of many of these developments and discussed them with his colleagues at the seminar.

Above all, the ideas of Heinrich Hertz and Ludwig Boltzmann on the foundations of physics strongly influenced him, not only at the methodological level, but also concerning his strong adherence to the mechanical reductionist point of view.

The lecture notes of courses in geometry taught by Hilbert in Königsberg illuminatingly exemplify the confluence of the various points mentioned in the preceding paragraphs. Central to this is his conception of geometry as a *natural* science, close in all respects to mechanics and the other physical disciplines, and opposed to arithmetic and other mathematical fields of enquiry. This was a traditional separation, adopted with varying degrees of commitment, among the German mathematicians (especially in Göttingen) since the time of Gauss. Even geometers like Moritz Pasch, who had stressed a thoroughly axiomatic approach in their presentations of projective geometry [20], would support such an empiricist view of geometry. In the introduction to a course taught in 1891, for instance, Hilbert expressed his views as follows:

> Geometry is the science dealing with the properties of space. It differs essentially from pure mathematical domains such as the theory of numbers, algebra, or the theory of functions. The results of the latter are obtained through pure thinking …The situation is completely different in the case of geometry. I can never penetrate the properties of space by pure reflection, much the same as I can never recognize the basic laws of mechanics, the law of gravitation or any other physical law in this way. Space is not a product of my reflections. Rather, it is given to me through the senses. ([5], p. 84)

The connection between this view and the axiomatic approach as a proper way to deal with this kind of sciences was strongly supported by the work of Hertz. Hilbert had announced another course in geometry for 1893, but for lack of students registered it was postponed until 1894. Precisely at this time, Hertz's *Principles of Mechanics* [13] was posthumous published, and Hilbert got enthusiastic notice of the book from his friend Minkowski. Minkowski had been in Bonn since 1885 where he came under the strong influence of Hertz, to the point that the latter became his main source of scientific inspiration ([15], p. 355). In the now famous introduction to his book, Hertz described physical theories as "pictures" (*Bilder*) that we form for ourselves of natural phenomena, and suggested three criteria to evaluate among several possible images of one and the same object: permissibility, correctness, and appropriateness. Permissibility corresponds very roughly to consistency, whereas correctness and appropriateness are closer to the kind of criteria that will appear later on in Hilbert's *Grundlagen der Geometrie* (*GdG* – see below).

In the lecture notes of his 1893–94 course, Hilbert referred once again to the natural character of geometry and explained the possible role of axioms in elucidating its foundations. As he had time to correct the notes, he now made explicit reference to Hertz's characterization of a "correct" scientific image (*Bild*) or theory. Thus Hilbert wrote ([5], p. 87):

> Nevertheless the origin [of geometrical knowledge] is in experience. The axioms are, as Hertz would say, images or symbols in our mind, such that consequents of the images are again images of the consequences, i.e., what we can logically deduce from the images is itself valid in nature.

Hilbert also pointed out the need of establishing the independence of the axioms of geometry, while alluding, once again, to the kind of demand stipulated by Hertz. Stressing the objective and factual character of geometry, Hilbert wrote:

> The problem can be formulated as follows: What are the necessary, sufficient, and mutually independent conditions that must be postulated for a system of things, in order that any of their properties correspond to a geometrical fact and, conversely, in order that a complete description and arrangement of all the geometrical facts be possible by means of this system of things.

The axioms of geometry and of physical disciplines, Hilbert said, "express observations of facts of experience, which are so simple that they need no additional confirmation by physicists in the laboratory."

The empirical character of geometry has its clear expression in the importance attributed to Gauss's measurement of the sum of angles of a triangle formed by three mountain peaks in Hannover. Hilbert found these measurements convincing enough to indicate the correctness of Euclidean geometry as a true description of physical space. Nevertheless, he envisaged the possibility that some future measurement would yield a different result. This example would arise very frequently in Hilbert's lectures on physics in years to come, as an example of how the axiomatic method should be applied in physics, where new empirical facts are often found by experiment. Faced with new such findings that seem to contradict an existing theory, the axiomatic analysis would allow making the necessary modifications on some of the basic assumptions of the theory, without however having to modify its essential logical structure. Hilbert stressed that the axiom of parallels is likely to be the one to be modified in geometry if new experimental discoveries would necessitate so. Geometry was especially amenable to a full axiomatic analysis only because of its very advanced stage of development and elaboration, and not because of any other specific, essential trait concerning its nature that would set it apart from other disciplines of physics. Thus, in a course on mechanics taught in 1899, the year of publication of *GdG*, he said:

> Geometry also [like mechanics] emerges from the observation of nature, from experience. To this extent, it is an *experimental science*.…But its experimental foundations are so irrefutably and so *generally acknowledged*, they have been confirmed to such a degree, that no further proof of them is deemed necessary. Moreover, all that is needed is to derive these foundations from a minimal set of *independent axioms* and thus to construct the whole edifice of geometry by *purely logical means*. In this way [i.e., by means of the axiomatic treatment] geometry is turned into a *pure mathematical* science. In mechanics it is also

> the case that all physicists recognize its most basic facts. But the *arrangement* of the basic concepts is still subject to changes in perception …and therefore mechanics cannot yet be described today as a *pure mathematical* discipline, at least to the same extent that geometry is. ([5], p. 90. Emphasis in the original)

Thus, at the turn of the century, Hilbert consolidated his view of the axiomatic method as a correct methodology to be applied, in parallel and with equal importance, to geometry and to all other physical disciplines. The publication of *GdG* helped spread his ideas very quickly and in strong association with geometry alone. But the idea of applying the same point of view to physics, although made known to the public only in the 1900 list of problems, was for him natural and evident from the outset. In his course of 1899, Hilbert devoted considerable effort to discussing the technical details of, as well as the logical and conceptual interrelations among, the main principles of analytical mechanics: the energy conservation principle, the principle of virtual velocities and the D'Alembert principle, the principles of straightest path and of minimal constraint, and the principles of Hamilton and Jacobi. All of this will appear prominently in Hilbert's later own elaboration of the program for the axiomatization of physics.

4. *Grundlagen der Geometrie*

Hilbert's *Grundlagen der Geometrie* embodied his first published, comprehensive presentation of an axiomatized mathematical discipline. Based on a course taught in the winter semester of 1898–99, it appeared in print in June of 1899. The declared aim of the book was to lay down a "simple" and "complete" system of "mutually independent" axioms, from which all known theorems of geometry might be deduced. The axioms were formulated for three systems of undefined objects named "points", "lines", and "planes", and they establish mutual relations that these objects must satisfy. The axioms were grouped into five categories: axioms of incidence, of order, of congruence, of parallels, and of continuity. From a purely logical point of view, the groups have no real significance in themselves. However, from the geometrical point of view they are highly significant, for they reflect Hilbert's actual conception of the axioms as an expression of spatial intuition: each group expresses a particular way that these intuitions manifest themselves in our understanding.

Hilbert's first requirement, that the axioms be independent, is the direct manifestation of the foundational concerns that guided his research. When analyzing independence, his interest focused mainly on the axioms of congruence, continuity and of parallels, since this independence would specifically explain how the various basic theorems of Euclidean and projective geometry are logically interrelated. This requirement had already appeared – albeit more vaguely formulated – in Hilbert's early lectures on geometry, as a direct echo of Hertz's demand for "appropriateness" of physical theories (i.e., the demand of "distinctness and simplicity" for the axioms of the theory). This time Hilbert also provided the tools to prove systematically the mutual independence among the individual axioms within the groups and among the various groups of axioms in the system. However, this was not for Hilbert an exercise in analyzing abstract relations among systems of axioms and their possible models. The motivation for enquiring about the mutual independence of the axioms remained, essentially, a geometrical one. For this reason, Hilbert's original system of axioms was not the most economical one from the logical point of view. Indeed, several mathematicians noticed quite soon that Hilbert's system of axioms, seen as a single collection rather than as a collection of five groups, contained a certain degree of redundancy ([19], [23]). Hilbert's own aim was to establish the interrelations among the groups of axioms, embodying the various manifestations of space intuition, rather than among individual axioms belonging to different groups.

The second one, simplicity is also related to Hertz's appropriateness. Unlike the other requirements, it did not become standard as part of the important mathematical ideas to which *GdG* eventually led. Through this requirement Hilbert wanted to express the desideratum that an axiom should contain "no more than a single idea." However, he did not provide any formal criterion to decide when an axiom is simple. Rather this requirement remained implicitly present in *GdG*, as well as in later works of Hilbert, as a merely aesthetic guideline that was never transformed into a mathematically controllable feature.

The idea of a complete axiomatic system became pivotal to logic after 1930 following the works of Gödel, and in connection with the finitist program for the foundations of arithmetic launched by Hilbert and his collaborators around 1920. This is not, however, what Hilbert had in mind in 1899, when he included a requirement under this name in the analysis presented in *GdG*. Rather, he was thinking of a kind of "pragmatic" completeness. In fact, what Hilbert was demanding here is that an adequate axiomatization of a mathematical discipline should allow for an actual derivation of *all* the theorems already known in that discipline. This was, Hilbert claimed, what the totality of his system of axioms did for Euclidean geometry or, if the axiom of parallels is ignored, for the so-called absolute geometry, namely that which is valid independently of the latter.

Also the requirement of consistency was to become of paramount importance thereafter. Still, as part of *GdG*, Hilbert devoted much less attention to it. For one thing, he did not even mention this task explicitly in the introduction to the book. For another, he devoted just two pages to discussing the consistency of his system in the body of the book. In fact, it is clear that Hilbert did not intend to give a direct proof of consistency of geometry here, but even an indirect proof of this fact does not explicitly appear in *GdG*, since a systematic treatment of the question implied a full discussion of the structure of the system of real numbers, which was not included. Rather, Hilbert suggested that it would suffice to show that the specific kind of synthetic geometry derivable from his axioms could be translated into the standard Cartesian geometry, if the axes are taken as representing the entire field of real numbers. Only in the second edition of *GdG*, published in 1903, Hilbert added an additional axiom, the so-called "axiom of completeness" (*Vollständigkeitsaxiom*), meant to ensure that, although infinitely many incomplete models satisfy all the other axioms, there is only one complete model that satisfies this last axiom as well, namely, the usual Cartesian geometry.

Hilbert's axiomatic analysis of geometry was not meant to encourage the possibility of choosing arbitrary combinations of axioms within his system, and of exploring their consequences. Rather, his analysis was meant to enhance our understanding of those systems with a more intuitive, purely geometrical significance – Euclidean geometry, above all – and that made evident the connection of his work with long-standing concerns of the discipline throughout the nineteenth century [8]. As already stressed, the definition of systems of abstract axioms and the kind of axiomatic analysis described above was meant to be carried out always retrospectively, and only for "concrete", *well-established and elaborated* mathematical entities.

The publication of the *Grundlagen* was followed by many further investigations into Hilbert's technical arguments, as well as by more general, methodological and philosophical discussions. One important such discussion appeared in the correspondence between Hilbert and Gottlob Frege. This interchange has drawn considerable attention of historians and philosophers, especially for the debate it contains between Hilbert and Frege concerning the nature of mathematical truth. But this frequently-emphasized issue is only one side of a more complex picture advanced by Hilbert in his letters. In particular, it is interesting to notice Hilbert's explanation to Frege, concerning the main motivations for undertaking his axiomatic analysis: the latter had arisen, in the first place, from difficulties Hilbert had encountered when dealing with *physical*, rather than mathematical theories. Echoing once again ideas found in the introduction to Hertz's textbook, and clearly having in mind the problematic conceptual situation of the kinetic theory of gases at the turn of the century, Hilbert stressed the need to analyze carefully the process whereby physicists continually add new assumptions to existing physical theories, without properly checking whether or not the former contradict the latter, or consequences of the latter. In a letter of December 29, 1899, Hilbert wrote to Frege:

> After a concept has been fixed completely and unequivocally, it is on my view completely illicit and illogical to add an axiom – a mistake made very frequently, especially by physicists. By setting up one new axiom after another in the course of their investigations, without confronting them with the assumptions they made earlier, and without showing that they do not contradict a fact that follows from the axioms they set up earlier, physicists often allow sheer nonsense to appear in their investigations. One of the main sources of mistakes and misunderstandings in modern physical investigations is precisely the procedure of setting up an axiom, appealing to its truth, and inferring from this that it is compatible with the defined concepts. One of the main purposes of my *Festschrift* was to avoid this mistake. ([9], p. 40)

In a different passage of the same letter, Hilbert commented on the possibility of substituting the basic objects of an axiomatically formulated theory by a different system of objects, provided the latter can be put in a one-to-one, invertible relation with the former. In this case, the known theorems of the theory are equally valid for the second system of objects. Concerning physical theories, Hilbert wrote:

> All the statements of the theory of electricity are of course valid for any other system of things which is substituted for the concepts magnetism, electricity, etc., provided only that the requisite axioms are satisfied. But the circumstance I mentioned can never be a defect in a theory [footnote: it is rather a tremendous advantage], and it is in any case unavoidable. However, to my mind, the application of a theory to the world of appearances always requires a certain measure of good will and tactfulness: e.g., that we substitute the smallest possible bodies for points and the longest possible ones, e.g., light-rays, for lines. At the same time, the further a theory has been developed and the more finely articulated its structure, the more obvious the kind of application it has to the world of appearances, and it takes a very large amount of ill will to want to apply the more subtle propositions of [the theory of surfaces] or of Maxwell's theory of electricity to other appearances than the ones for which they were meant …([9], p. 41)

Hilbert's letters to Frege help understanding the importance of the link between physical and mathematical theories on the development of his axiomatic point of view. The latter clearly did not involve either an empty game with arbitrary systems of postulates nor a conceptual break with the classical, nineteenth-century entities and problems of mathematics and empirical science. Rather it sought after an improvement in the mathematician's understanding of the latter. This motto was to guide much of Hilbert's incursions into several domains of physics over the years to come.

5. Physics and the 1900 list of problems

In the introductory section of his Paris talk, Hilbert stressed the important role he accorded to empirical motivations as a fundamental source of nourishment for what he described as a "living organism", in which mathematics and the physical sciences appear tightly interrelated. The empirical motivations underlying mathematical ideas, Hilbert said, should by no means be taken as opposed to rigor. On the contrary, contrasting an "opinion occasionally advocated by eminent men", Hilbert insisted that the contemporary quest for rigor in analysis and arithmetic should in fact be *extended to both geometry and the physical sciences*. He was alluding here, most probably, to Kronecker and Weierstrass, and the Berlin purist tendencies that kept geometry and applications out of their scope of interest. Rigorous methods are often simpler and easier to understand, Hilbert said, and therefore, a more rigorous treatment would only perfect our understanding of these topics, and at the same time would provide mathematics with ever new and fruitful ideas. In explaining why rigor should not be sought only within analysis, Hilbert actually implied that this rigor should actually be pursued in axiomatic terms. He thus wrote:

> Such a one-sided interpretation of the requirement of rigor would soon lead to the ignoring of all concepts arising from geometry, mechanics and physics, to a stoppage of the flow of new material from the outside world, and finally, indeed, as a last consequence, to the rejection of the ideas of the continuum and of irrational numbers. But what an important nerve, vital to mathematical science, would be cut by rooting out geometry and mathematical physics! On the contrary I think that wherever mathematical ideas come up, whether from the side of the theory of knowledge or in geometry, or from the theories of natural or physical science, the problem arises for mathematics to investigate the principles underlying these ideas and to establish them upon a simple and complete system of axioms, so that the exactness of the new ideas and their applicability to deduction shall be in no respect inferior to those of the old arithmetical concepts. (Quoted from [12], p. 245)

Using a rhetoric reminiscent of Volkmann's work, Hilbert described the development of mathematical ideas as an ongoing, dialectical interplay between the two poles of thought and experience. He also added an idea that was of central importance to Göttingen scientists for many decades, namely, the conception of the "pre-established harmony" between mathematics and nature ([21]). The importance of investigating the foundations of mathematics does not appear as an isolated concern, but rather as an organic part of the manifold growth of the discipline in several directions. Hilbert thus said:

> Indeed, the study of the foundations of a science is always particularly attractive, and the testing of these foundations will always be among the foremost problems of the investigator …[But] a thorough understanding of its special theories is necessary for the successful treatment of the foundations of the science. Only that architect is in the position to lay a sure foundation for a structure who knows its purpose thoroughly and in detail. (Quoted from [12], p. 258)

The first two problems in Hilbert's list are Cantor's continuum hypothesis and the compatibility of the axioms of arithmetic. In formulating the second problem on his list, Hilbert stated more explicitly than ever before, that among the tasks related to investigating an axiomatic system, proving its consistency would be the most important one. Yet, Hilbert was still confident that this would be a rather straightforward task, easily achievable "by means of a careful study and suitable modification of the known methods of reasoning in the theory of irrational numbers." Clearly Hilbert meant his remarks in this regard to serve as an argument against Kronecker's negative reactions to unrestricted use of infinite collections in mathematics, and therefore he explicitly asserted that a consistent system of axioms could prove the existence of higher Cantorian cardinals and ordinals. Hilbert's assertion is actually the first published mention of the paradoxes of Cantorian set theory, which here were put forward with no special fanfare ([7], p. 301). He thus established a clear connection between the two first problems on his list through the axiomatic approach. Still, Hilbert was evidently unaware of the difficulties involved in realizing this point of view, and, more generally, he most likely had no precise idea of what an elaborate theory of systems of axioms would involve. On reading the first draft of the Paris talk, several weeks earlier, Minkowski understood at once the challenging implications of Hilbert's view, and he hastened to write to his friend:

> In any case, it is highly original to proclaim as a problem for the future, one that mathematicians would think they had already completely possessed for a long time, such as the axioms for arithmetic. What might the many laymen in the auditorium say? Will their respect for us grow? And you will also have a though fight on your hands with the philosophers. ([22], p. 129)

Frege's reaction to the *GdG* proved Minkowski's concern to be justified, as his main criticism referred to the status of axioms as implicit definitions.

The next three problems in the list are directly related with geometry and, although not explicitly formulated in axiomatic terms, they address the question of finding the correct relationship between specific assumptions and specific, significant geometrical facts. The fifth problem, for instance, relates to the question of the foundations of geometry as it had evolved over the last third of the nineteenth century along two parallel paths. On the one hand, there was the age-old tradition of elementary synthetic geometry, where the question of foundations more naturally arises in axiomatic terms. On the other hand, there was the tradition associated with the Helmholtz–Lie problem, that derived directly from the work of Riemann and that had a more physically-grounded orientation connected with the question of spaces that admit the free mobility of rigid bodies. Whereas Helmholtz had only assumed *continuity* as underlying the motion of rigid bodies, in applying his theory of groups of transformations to this problem, Lie was also assuming the *differentiability* of the functions involved. Hilbert's work on the foundations of geometry, especially in the context that led to *GdG*, had so far been connected with the first of these two traditions, while devoting much less attention to the second one. Now in his fifth problem, he asked whether Lie's conditions, rather than assumed, could actually be deduced from the group concept together with other geometrical axioms.

As a mathematical problem, the fifth one led to interesting, subsequent developments. Not long after his talk, in November 18, 1901, Hilbert himself proved that, in the plane, the answer is positive, and he did so with the help of a then innovative, essentially topological, approach [14]. That the answer is positive in the general case was satisfactorily proved only in 1952 ([10], [18]). The inclusion of this problem in the 1900 list underscores the actual scope of Hilbert's views over the question of the foundations of geometry and over the role of axiomatics. Hilbert suggested here the pursuit of an intricate kind of conceptual clarification involving assumptions about motion, differentiability and symmetry, such as they appear intimately interrelated in the framework of a well-elaborate mathematical theory, namely, that of Lie. This quest, that also became typical of the spirit of Hilbert's axiomatic involvement with physical theories, suggests that his foundational views on geometry were very broad and open-ended, and did not focus on those aspects related with the synthetic approach to geometry. In particular, the fifth problem emphasizes the prominent role that Hilbert assigned to physical considerations in his approach to geometry. In the long run, this aspect of Hilbert's view resurfaced at the time of his involvement with GTR ([5], Ch. 7–8). In its more immediate context, however, it makes the passage from geometry to the sixth problem appear as a natural one within the list.

Indeed, if the first two problems in the list show how the ideas deployed in *GdG* led in one direction towards foundational questions in arithmetic, then the fifth problem suggests how they also naturally led, in a different direction, to Hilbert's call for the axiomatization of physical science in the sixth problem. The problem was thus formulated as follows:

> The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics. (Quoted in [12], p. 258)

As examples of what he had in mind Hilbert mentioned several existing and well-known works: the fourth edition of Mach's *Die Mechanik in ihrer Entwicklung*, Hertz's *Prinzipien*, Boltzmann's 1897 *Vorlesungen Über die Principien der Mechanik*, and also Volkmann's 1900 *Einführung in das Studium der theoretischen Physik*. Boltzmann's work offered a good example of what axiomatization would offer, as he had indicated, though only schematically, that limiting processes could be applied, starting from an atomistic model, to obtain the laws of motion of continua. Hilbert thought it convenient to go in the opposite direction also, i.e., to derive the laws of motions of rigid bodies by limiting processes, starting from a system of axioms that describe space as filled with continuous matter in varying conditions. Thus one could investigate the equivalence of different systems of axioms, an investigation that Hilbert considered to be of the highest theoretical importance.

This is one of the few places where Hilbert emphasized Boltzmann's work over Hertz's in this regard, and this may give us the clue to the most immediate trigger that was in the back of Hilbert's mind when he decided to include this problem in the list. Indeed, Hilbert had met Boltzmann several months earlier in Munich, where the latter gave a talk on recent developments in physics. Boltzmann had not only discussed ideas connected with the task that Hilbert was now calling for, but he also adopted a rhetoric that seems to have appealed very much to Hilbert. In fact, Boltzmann had suggested that one could follow up the recent history of physics with a look at future developments. Nevertheless, he said, "I will not be so rash as to lift the veil that conceals the future" ([2], p. 79). Hilbert, on the contrary, opened the lecture by asking precisely, "who among us would not be glad to lift the veil behind which the future lies hidden" and the whole trust of his talk implied that he, the optimistic Hilbert, was helping the mathematical community to do so.

Together with the well-known works on mechanics referred to above, Hilbert also mentioned a recent work by the Göttingen actuarial mathematician Georg Bohlmann on the foundations of the calculus of probabilities [1]. The latter was important for physics, Hilbert said, for its application to the method of mean values and to the kinetic theory of gases. Hilbert's inclusion of the theory of probabilities among the main physical theories whose axiomatization should be pursued has often puzzled readers of this passage. The notes of a course taught in 1905 on the axiomatic method show that this was a main point in Hilbert's views on physics because of the use of probabilities also in insurance mathematics and in problems of observational error calculation in astronomy. It is also remarkable that Hilbert did not mention electrodynamics among the physical disciplines to be axiomatized, even though the second half of the *Gauss–Weber Festschrift*, where Hilbert's *GdG* was published, contained a parallel essay by Wiechert on the foundations of electrodynamics. At any rate, Wiechert's presentation was by no means axiomatic, in any sense of the term. On the other hand, the topics addressed by Wiechert would start attracting Hilbert's attention over the next years, at least since 1905.

This sixth problem is not really a problem in the strict sense of the word, but rather a general task for whose complete fulfillment Hilbert set no clear criteria. Thus, Hilbert's detailed account in the opening remarks of his talk as to what a meaningful problem in mathematics is, and his stress on the fact that a solution to a problem should be attained in a finite number of steps, does not apply in any sense to the sixth one. On the other hand, the sixth problem has important connections with three other problems on Hilbert's list: the nineteenth ("Are all the solutions of the Lagrangian equations that arise in the context of certain typical variational problems necessarily analytic?"), the twentieth (dealing with the existence of solutions to partial differential equations with given boundary conditions), closely related to the nineteenth and at the same time to Hilbert's long-standing interest on the Dirichlet Principle, and, finally, the twenty-third (an appeal to extend and refine the existing methods of variational calculus). Like the sixth problem, the latter two are general tasks rather than specific mathematical problems with a clearly identifiable, possible solution. All these three problems are also strongly connected to physics, though unlike the sixth, they are also part of mainstream, traditional research concerns in mathematics. In fact, their connections to Hilbert's own interests are much more perspicuous and, in this respect, they do not raise the same kind of historical questions that Hilbert's interest in the axiomatization of physics does.

A balanced assessment of the influence of the problems on the development of mathematics throughout the century must take into account not only the intrinsic importance of the problems, but also the privileged institutional role of Göttingen in the mathematical world with the direct and indirect implications of its special status. However, if Hilbert wished to influence the course of mathematics over the coming century with his list, then it is remarkable that his own career was only very partially shaped by it. Part of the topics covered by the list belonged to his previous domains of research, while others belonged to domains where he never became active. On the contrary, domains that he devoted much effort to over the next years, such as the theory of integral equations, were not contemplated in the list. In spite of the enormous influence Hilbert had on his students, the list did not become a necessary point of reference of preferred topics for dissertations. To be sure, some young mathematicians, both in Göttingen and around the world, did address problems on the list and sometimes came up with important mathematical achievements that helped launch their own international careers. But this was far from the only way for talented young mathematicians to reach prominence in or around Göttingen. But, ironically, the sixth problem, although seldom counted among the most influential of the list, can actually be counted among those that received greater attention from Hilbert himself and from his collaborators and students over the following years.

6. Concluding remarks

For all its differences and similarities with other problems on the list, the important point that emerges from the above account is that the sixth problem was in no sense disconnected from the evolution of Hilbert's early axiomatic conception at its very core. Nor was it artificially added in 1900 as an afterthought about the possible extensions of an idea successfully applied in 1899 to the case of geometry. Rather, Hilbert's ideas concerning the axiomatization of physical science arose simultaneously with his increasing enthusiasm for the axiomatic method and they fitted naturally into his overall view of pure mathematics, geometry and physical science – and the relationship among them – by that time.

From 1900 on, the idea of axiomatizing physical theories was a main thread that linked much of Hilbert's research and teaching. Hilbert taught every semester at least one course dealing with a physical discipline, and by the end of his career he had covered most of the important fields that were at the cutting edge of physics, currently attracting the best research efforts of young and promising minds (see the appendix to this article). The axiomatic point of view provided a unifying methodology from which to approach many of the topics in which Hilbert became interested. In 1905 he taught a course on the axiomatic method where he presented for the first time a panoramic view of various physical disciplines from an axiomatic perspective: mechanics, thermodynamics, probability calculus, kinetic theory, insurance mathematics, electrodynamics, psychophysics. The variety of physical topics pursued only grew over the years. The extent of the influence of Hilbert's ideas on physics on contemporary research is a more complex question that cannot be discussed here for lack of space. Still, it is relevant to quote here an account of Hilbert's ideas as described by the physicist on whom Hilbert's influence became most evident, Max Born. On the occasion of Hilbert's sixtieth birthday, at a time when he was deeply involved together with Bernays in the technical difficulties raised by the finitist program, Born wrote the following words:

> The physicist set outs to explore how things are in nature; experiment and theory are thus for him only a means to attain an aim. Conscious of the infinite complexities of the phenomena with which he is confronted in every experiment, he resists the idea of considering a theory as something definitive. He therefore abhors the word "Axiom", which in its usual usage evokes the idea of definitive truth. The physicist is thus acting in accordance with his healthy instinct, that dogmatism is the worst enemy of natural science. The mathematician, on the contrary, has no business with factual phenomena, but rather with logic interrelations. In Hilbert's language the axiomatic treatment of a discipline implies in no sense a definitive formulation of specific axioms as eternal truths, but rather the following methodological demand: specify the assumptions at the beginning of your deliberation, stop for a moment and investigate whether or not these assumptions are partly superfluous or contradict each other. ([3])

The development of physics from the beginning of the century, and especially *after* 1905, brought many surprises that Hilbert could not have envisaged in 1900 or even when he lectured at Göttingen on the axioms of physics in 1905; yet, Hilbert was indeed able to accommodate these new developments to the larger picture of physics afforded by his program for axiomatization. In fact, some of his later contributions to mathematical physics, particularly his contributions to GTR, came by way of realizing the vision embodied in this program.

7. Appendix: Hilbert's Göttingen courses on physics (and related fields): 1895–1927

For an explanation on the sources used for compiling this list, see [5], p. 450 (WS = Winter Semester, SS = Summer Semester, HS = Special Autumn [Herbst] Semester).

WS 1895/96	Partial Differential Equations
SS 1896	Ordinary Differential Equations
SS 1898	Mechanics
SS 1899	Variational Calculus
WS 1900/01	Partial Differential Equations
SS 1901	Linear Partial Differential Equations
WS 1901/02	Potential Theory

SS 1902	Selected Topics in Potential Theory
WS 1902/03	Continuum Mechanics - Part I
SS 1903	Continuum Mechanics - Part II
WS 1903/04	Partial Differential Equations
WS 1904/05	Variational Calculus
SS 1905	Logical Principles of Mathematical Thinking (and of Physics)
SS 1905	Integral Equations
WS 1905/06	Partial Differential Equations
WS 1905/06	Mechanics
SS 1906	Integral Equations
WS 1906/07	Continuum Mechanics
SS 1907	Differential Equations
WS 1909/10	Partial Differential Equations
SS 1910	Selected Chapters in the Theory of Partial Differential Equations
WS 1910/11	Mechanics
SS 1911	Continuum Mechanics
WS 1911/12	Statistical Mechanics
SS 1912	Radiation Theory
SS 1912	Ordinary Differential Equations
SS 1912	Mathematical Foundations of Physics
WS 1912/13	Molecular Theory of Matter
WS 1912/13	Partial Differential Equations
WS 1912/13	Mathematical Foundations of Physics
SS 1913	Foundations of Mathematics (and the axiomatization of Physics)
SS 1913	Electron Theory
WS 1913/14	Electromagnetic Oscillations
WS 1913/14	Analytical Mechanics
WS 1913/14	Exercises in Mechanics (together with H. Weyl)
SS 1914	Statistical Mechanics
SS 1914	Differential Equations
WS 1914/15	Lectures on the Structure of Matter
SS 1915	Structure of Matter (Born's Theory of Crystals)
WS 1915/16	Differential Equations
SS 1916	Partial Differential Equations
SS 1916	Foundations of Physics I (General Relativity)
WS 1916/17	Foundations of Physics II (General Relativity)
SS 1917	Electron Theory
SS 1918	Ordinary Differential Equations
WS 1918/19	Space and Time
WS 1918/19	Partial Differential and Integral Equations
HS 1919	Nature and Mathematical Knowledge
WS 1920	Mechanics
SS 1920	Higher Mechanics and the New Theory of Gravitation

WS 1920/21	Mechanics and the New Theory of Gravitation
SS 1921	Einstein's Gravitation Theory. Basic Principles of the Theory of Relativity
SS 1921	On Geometry and Physics
SS 1922	Statistical Mechanics
WS 1922/23	Mathematical Foundations of Quantum Theory
WS 1922/23	Knowledge and Mathematical Thought
WS 1922/23	Knowledge and Mathematical Thought
SS 1923	Our Conception of Gravitation and Electricity
WS 1923/24	On the Unity of Science
SS 1924	Mechanics and Relativity Theory
WS 1926/27	Mathematical Methods of Quantum Theory
SS 1930	Mathematical Methods of Modern Physics
WS 1930/31	Nature and Thought
WS 1931/32	Philosophical Foundations of Modern Natural Science

References

[1] Bohlmann, G., Ueber Versicherungsmathematik. In *Über angewandte Mathematik und Physik in ihrer Bedeutung für den Unterricht an den höheren Schulen* (ed. by F. Klein & E. Riecke), Teubner, Leipniz, Berlin 1900, 114–145.

[2] Boltzmann, L., Über die Entwicklung der Methoden der theoretischen Physik in neuerer Zeit (1899). In L. Boltzmann *Populäre Schriften*, J. A. Barth, Leipzig 1905, 198–277.

[3] Born, M., Hilbert und die Physik. *Die Naturwissenschaften* 10 (1922), 88–93. (Reprint in Born. M., *Ausgewählte Abhandlungen*, Vol. 2, Vandenhoek & Ruprecht, Göttingen 1963, 584–598.)

[4] Brush, S. G., *The Kind of Motion we Call Heat - A History of the Kinetic Theory of Gases in the 19th Century*. North Holland Publishing House, Amsterdam, New York, Oxford 1976.

[5] Corry, L., *David Hilbert and the Axiomatization of Physics (1898–1918)*: From *Grundlagen der Geometrie* to *Grundlagen der Physik.* Archimedes: New Studies in the History and Philosophy of Science and Technology 10, Kluwer Academic Publishers, Dordrecht 2004.

[6] Dieudonné, J., Les méthodes axiomatiques modernes et les fondements des mathématiques. In *Les grands Courants de la Pensée Mathématique* (ed. by F. Le Lionnais), Blanchard, Paris 1962, 443–555.

[7] Ferreirós, J., *Labyrinths of Thought. A History of Set Theory and its Role in Modern Mathematics*. Sci. Networks Hist. Stud. 23, Birkhäuser, Boston 1999.

[8] Freudenthal, H., Zur Geschichte der Grundlagen der Geometrie. Zugleich eine Besprechung der 8. Auflage von Hilberts 'Grundlagen der Geometrie'. *Nieuw Archief voor Wiskunde* **4** (1957), 105–142.

[9] Gabriel, G. et al. (eds.), *Gottlob Frege - Philosophical and Mathematical Correspondence*. The University of Chicago Press, Chicago 1980.

[10] Gleason, A., Groups without Small Subgroups. *Ann. Math.* **56** (1952), 193–212.

[11] Gnedenko, J., Zum sechsten Hilbertschen Problem. In *Die Hilbertsche Probleme* (ed. by P. Alexandrov), Ostwalds Klassiker der exakten Wissenschaften 252, Leipzig 1979, 144–147.

[12] Gray, J. J., *The Hilbert Challenge*. Oxford University Press, New York 2000.

[13] Hertz, H., *Die Prinzipien der Mechanik in neuem Zusammenhange dargestellt*. Leipzig 1984.

[14] Hilbert, D., Über die Grundlagen der Geometrie. *Math. Ann.* **56** (1902), 233–241.

[15] Hilbert, D., Hermann Minkowski. *Math. Ann.* **68** (1910), 445–471.

[16] Hilbert, D., *Natur und Mathematisches Erkennen: Vorlesungen, gehalten 1919-1920 in Göttingen. Nach der Ausarbeitung von Paul Bernays*. Edited and with an English introduction by David E. Rowe, Birkhäuser, Basel 1992.

[17] Hilbert, D., Über meine Tätigkeit in Göttingen. In *Hilbert: Gedenkband* (ed. by K. Reidemeister). Springer-Verlag, Berlin, Heidelberg, New York 1971, 79–82.

[18] Montgomery, D., Zippin, L., Small Subgroups of Finite-dimensional Groups. *Ann. Math.* **56** (1952), 213–241.

[19] Moore. E. H., Projective Axioms of Geometry. *Trans. Amer. Math. Soc.* **3** (1902), 142–158.

[20] Pasch, M., *Vorlesungen über neuere Geometrie*. Teubner, Leipzig 1882.

[21] Pyenson, L., Relativity in Late Wilhelmian Germany: the Appeal to a Pre-established Harmony Between Mathematics and Physics. In L. Pyenson *The Young Einstein: The Advent of Relativity*, Adam Hilger Ltd., Bristol, Boston 1985, 137–157.

[22] Rüdenberg L., Zassenhaus, H., *Hermann Minkowski - Briefe an David Hilbert*. Springer-Verlag, Berlin, New York 1973.

[23] Schur, F., Über die Grundlagen der Geometrie. *Math. Ann.* **55** (1901), 265–292.

[24] Weyl, H., David Hilbert and his Mathematical Work, *Bull. Amer. Math. Soc.* **50** (1944), 612–654.

[25] Wightman, A. S., Hilbert's Sixth Problem: Mathematical Treatment of the Axioms of Physics. In *Mathematical Developments Arising from Hilbert Problems* (ed. by F. E. Browder), Symposia in Pure Mathematics 28, Amer. Math. Soc., Providence, RI, 1976, 147–240.

[26] Yandell, B. H., *The Honors Class: Hilbert's Problems and Their Solvers*. AK Peters, Natick, MA, 2002.

Cohn Institute for History and Philosophy of Science, Tel-Aviv University, Tel-Aviv 69978, Israel

E-mail: corry@post.tau.ac.il