Decidability

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Monday 10^{th} April, 2017

Decision Problems

We may be interested in following questions:

> "Is a number number perfect square?"
> "Is number prime?"
> "Does a graph has cycle?"
> "Does the computation of TM halt before 25th transition?"

Each of these general question describe a decision problem. A decision problem PP is a set of *related questions* p_i, each of which has es/no answer. For example:

p_0: Is 0 a perfect square?

p_1 Is 1 a perfect square?

p_2 is 2 a perfect square?

...

Each of the p_i is an instance of the problem P. The solution of a decision problem P is an algorithm that determines the answer of every question $p \in P$. A decision problem is decidable, if it has a solution.

Decision Problems

An algorithm that solves decision problem should be:

- Complete: correct answer is given for every problem instance
- Mechanistic: finite sequence of instructions, each can be carried out without requirement of insight, ingenuity, or guesswork.
- Deterministic: With identical input, the same computation is carried.

A procedure having the properties of complete, Mechanistic, and deterministic, is called *effective procedure*. A standard TM is an effective algorithm if it is, Mechanistic, deterministic, and complete. However, it is complete only if, it halts on every input.

Decidable languages

Decidable language-3

- $A_{EQDFA} = \{\langle A, B\rangle |\ A \text{ and } B \text{ are DFAs and } L(A) = L(B)\ \}$
- Equivalence problem: Test whether two DFAs recognize the same language.
- Theorem: A_{EQDFA} is decidable languages
- $F =$ "On input $\langle A, B\rangle$, where A and B are DFAs:
  1. construct DFA $L(C) = (L(A) \cap \overline{L(B)}) \cup (L(B) \cap \overline{L(A)})$ $(A = B \Rightarrow C = \phi)$
  2. Run $TM\ T$ for deciding A_{EQDFA} on input $\langle C\rangle$
  3. if T accepts, Accept; otherwise reject. "

Acceptance problem:A_{TM}

Input: A TM's description $\langle M\rangle$ and a string w for input to M.

Output: Yes/No indicating if M eventually enters q_{accept} on input w.

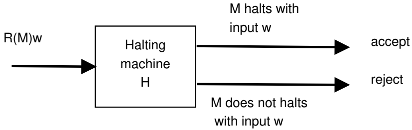
- Acceptance of language consisting of tuples: $\langle M, w\rangle$, $A_{TM} = \{\langle M, w\rangle | M \text{ is a Turing description and } M \text{ accepts input } w\ \}$. Is A_{TM} Turing recognizable?
- Defn. Turing Recognizable: A language A_{TM} is "Turing recognizable" if there exists a TM M such that for all w:

- If $\langle M, w\rangle \in A_{TM}$ then M eventually enter q_{accept}
- If $\langle M, w\rangle \notin A_{TM}$ then M eventually enter q_{regect} or M loops for ever.

Halting Problem

- Theorem: A_{TM} is Turing recognizable.
- $U =$ "On input $\langle M, w \rangle$, where M is a TM and w is a string:
  1. Simulate M on input w
  2. If M ever enters accepts state, then U accepts; if M enters its reject state, U rejects"
  - U is universal TM
  - U keeps looping if M neither accepts or rejects
  - However, if $U \equiv M, A_{TM}$ is unsolvable (i.e.,undecidable)
- A problem is decidable if some TM decides (solves) it.

  Halting problem: Given a TM M and input string $\langle W, \langle m \rangle \rangle$, decide whether M halt on $\langle \langle M \rangle, w \rangle$?
- The instance of the problem is : $e_n(M)e_n(w)$. Halting $= \{e_n(M)e(w) | M$ halts on w $\}$ is *not recursive*.



Undecidability of A_{TM}

Proof by contradiction:

- Assume that $\exists$ some *TM H* that decides A_{TM}. That is, *H accepts* if *M* accepts *w*, and *H rejects* if *M rejects w*.
- Now we construct a TM *D* with H as subroutine. This calls H and determine what M does when the input to M is its own description $< M >$. However, after determining this, it outputs the opposite. That is, it rejects if M accepts, and vice-versa. Call this as H'.
- Define $D(< M >) =$

1 Construct a *TM* D (having input $< M >$) that outputs the opposite of the result of simulating *H* on input $\langle\langle M\rangle, M\rangle$.

2 Output the opposite of what H outputs, i.e., if H accepts, then reject, and if H rejects, then accept.

- The above can be rewritten as:
- If *M* accepts its own description $< M >$, then
  $H(< M >)$ accepts and $\therefore D(< M >)$ rejects
  If *M* rejects its own description $< M >$, then
  $H(< M >)$ rejects and $\therefore D(< M >)$ accepts
- What happens if we run D on its own description $< D >$?
- From above: (substitute D for M), we have (see next slide)

Proving Undecidability of A_{TM}

If D accepts $< D >$:

$H(D, < D >)$ accepts and $D < D >$ rejects

If D rejects $< D >$:

$H(D, < D >)$ rejects and $D < D >$ accepts

- Which can be further simplified:

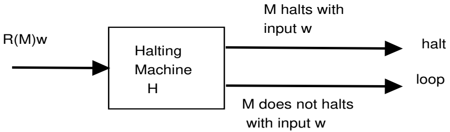
If D accepts $< D >$:

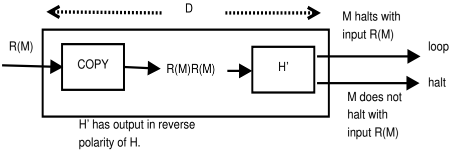
$D < D >$ rejects

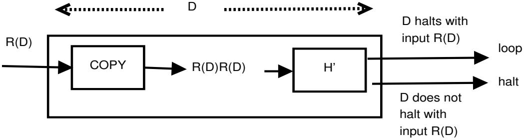
If D rejects $< D >$:

$D < D >$ accepts

- Hence, whatever is done, it must do the opposite. So there is a contradiction. So, D cannot exist. But, if H exists, we know how to make D. H cannot exist; so there is no TM that decides A_{TM}.







if D halts with input R(D) then loop
if D does not halt with input R(D) then halt

A different approach for A_{TM} as undecidable

- Preceding proof uses self-reference and diagonalization.
- To obtain table for diagonalization argument, consider that every $v \in \{0,1\}^*$ represent a *TM*. If v does not have form *R(M)*, a one-state TM with no transition is assigned to v. Thus, *TMs* can be listed as $M_0, M_1, M_2, M_3, M_4, M_5, M_6, M_7, \ldots$ corresponding to $\varepsilon, 0, 1, 00, 01, 10, 11, 000$.
- Consider a table that lists TMs along the horizontal and vertical axes. The *i, jth* entry in table is:

$$\begin{cases} 1 & \text{if } M_i \text{ halts when run with input } R(M_j) \\ 0 & \text{if } M_i \text{ does not halt when run with input } R(M_j) \end{cases}$$

Diagonal elements are answers to the self-referential questions: Does M_i halt when run with itself?

Undecidable, decidable, recognizable, Unrecognizable:

- A_{CFG} is decidable
- A_{TM} is undecidable
- $L \in P(\Gamma^*)$ is unrecognizable, where $P(\Gamma^*)$ is uncountable
- $\bar{A}_{TM} = \{w |$ M is a TM and M does not accept w$\}$